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The Navier-Stokes equations on manifolds with boundary ☆

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Abstract

We consider the motion of an incompressible viscous fluid on a compact Riemannian manifold M with boundary. The motion on M is modeled by the incompressible Navier-Stokes equations, and the fluid is subject to pure or partial slip boundary conditions of Navier type on $\partial\mathsf{M}$. We establish existence and uniqueness of strong as well as weak (variational) solutions for initial data in critical spaces. Moreover, we show that the set of equilibria consists of Killing vector fields on M that satisfy corresponding boundary conditions, and we prove that all equilibria are (locally) stable. In case M is two-dimensional we show that solutions with divergence free initial condition in $L_2(\mathsf{M}; T\mathsf{M})$ exist globally and converge to an equilibrium exponentially fast.

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1. Introduction

Suppose that M is a compact, smooth, connected and oriented n-dimensional Riemannian manifold with boundary $\Sigma = \partial\mathsf{M}$. It follows that Σ is a compact, smooth, orientable $(n-1)$-dimensional manifold. Σ is then provided with outward orientation with respect to M. Let $(\cdot|\cdot)_g$ denote the Riemann metric on M. In the sequel, we also use the notation $(\cdot|\cdot)_g$ for the induced Riemann metric on Σ. We will study the motion of an incompressible viscous fluid on M, modeled by the *surface Navier-Stokes equations* with *Navier boundary conditions* which can be stated as follows

$$\begin{cases} \varrho\,(\partial_t u + \nabla_u u) - 2\mu_s \mathsf{div}\, D(u) + \mathsf{grad}\,\pi = 0 & \text{on} \quad \mathsf{M}, \\ \mathsf{div}\, u = 0 & \text{on} \quad \mathsf{M}, \\ \alpha u + \mathcal{P}_\Sigma\left((\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) = 0 & \text{on} \quad \Sigma, \\ (u|\nu_\Sigma)_g = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0 & \text{on} \quad \mathsf{M}. \end{cases} \tag{1.1}$$

Here, the unknowns are the fluid velocity u and the fluid pressure π. $\varrho > 0$ is the (constant) density, $\mu_s > 0$ is the surface shear viscosity, ν_Σ is the outward unit normal field of Σ, while $\mathcal{P}_\Sigma$ is the orthogonal projection onto the tangent bundle of Σ, and the constant $\alpha \geq 0$ is a given friction parameter. In the following, we assume without loss of generality that $\varrho = 1$.

Moreover, $\nabla_u v$ denotes the covariant derivative induced by the Levi-Civita connection of M for given tangent vectors u, v, and $D(u) := \frac{1}{2}(\nabla u + [\nabla u]^{\mathsf{T}})^\sharp$ denotes the deformation tensor (a definition of the operator $^\sharp$ is provided in Appendix A), given in local coordinates by

$$D(u) = \frac{1}{2}\left(g^{jk}u^i_{|k} + g^{ik}u^j_{|k}\right)\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},$$

with $u^i_{|k}$ being covariant derivatives, that is,

$$u^i_{|k} = \partial_k u^i + \Gamma^i_{k\ell}u^\ell \quad \text{for} \quad u = u^i\frac{\partial}{\partial x^i}.$$

Here and throughout this article, we are using the Einstein summation convention, indicating that terms with repeated indices are added.

We note here that

$$D_u := \frac{1}{2}\left(\nabla u + [\nabla u]^{\mathsf{T}}\right)$$

is a $(1,1)$-tensor, while $D(u)$ is a $(2,0)$-tensor. In case $\mathsf{div}\, u = 0$, it is well-known, see for instance [35, Lemma 2.1], that

$$2\,\mathsf{div}\, D(u) = \Delta_{\mathsf{M}} u + \mathsf{Ric}^\sharp u, \tag{1.2}$$

where Δ_{M} denotes the (negative) Bochner Laplacian (sometimes also called the connection Laplacian), and $\mathsf{Ric}^\sharp$ is the Ricci $(1,1)$-tensor. In local coordinates, these operators are expressed by

$$\Delta_{\mathsf{M}} u = g^{ij}(\nabla_i \nabla_j - \Gamma^k_{ij}\nabla_k)u, \quad \mathsf{Ric}^\sharp u = R^i_j u^j \frac{\partial}{\partial x^i} := g^{ik} R_{kj} u^j \frac{\partial}{\partial x^i}, \tag{1.3}$$

with $\nabla_j = \nabla_{\frac{\partial}{\partial x^j}}$ being covariant derivatives, and where $\mathsf{Ric} = R_{ij}\,dx^i \otimes dx^j$ is the usual Ricci $(0,2)$-tensor. More details are given in Appendix A.

$(1.1)_3$ and $(1.1)_4$ is termed the *pure slip* boundary condition in case $\alpha = 0$, or *partial slip* boundary condition in case $\alpha > 0$.

In case $\mathsf{M} = \mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0,\infty)$, the boundary conditions for $u = (u^1, \dots, u^n)$ on $\Sigma = \mathbb{R}^{n-1}$ result in

$$u^n = 0, \qquad (\partial_j u^n + \partial_n u^j) - \alpha u^j = 0, \quad j = 1, \dots, n-1,$$

which, taking into account the relation $u^n = 0$, further reduce to

$$u^n = 0, \qquad \partial_n u^j - \alpha u^j = 0, \quad j = 1, \dots, n-1.$$

This implies $u^j(x', x^n) \approx (1 + \alpha x^n) u^j(x', 0)$ for small $x^n > 0$, showing the friction effect on Σ for tangential velocity components in case $\alpha > 0$.

The topic of fluids on surfaces and Riemannian manifolds has recently attracted attention by numerous authors, see for instance [6,19,23,25,27,30,33,35,37] and the references contained in these publications.

One application concerns the modeling of emulsion and biological membranes, see [38]. In addition, (1.1) may be considered as a model for the motion of a fluid on a planet's surface that is covered by water and landmasses (while the effect of Coriolis forces is being ignored).

The main results in this manuscript establish existence, uniqueness, and qualitative properties of strong as well as weak (variational) solutions to (1.1). The expression '(variational) weak solutions' is used here to distinguish our solutions from the class of Leray-Hopf weak solutions. Our approach is based on the method of L_p-L_q maximal regularity in time weighted spaces, see for instance [28].

In Sections 3 and 4, we demonstrate that the Stokes operator associated with (1.1) admits a bounded H^∞-calculus with angle $< \pi/2$ (a property that implies maximal regularity) in $L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$ as well as in $H^{-1}_{q,\sigma}(\mathsf{M}; T\mathsf{M})$. This property opens up the way to obtain unique solutions to (1.1) for initial data in critical spaces, as shown in Section 5, Theorem 5.1, Corollary 5.2, and Remark 5.3.

In Section 6, we show that the set of equilibria of (1.1) consists exactly of all Killing fields on M which satisfy the boundary conditions imposed on solutions, see Proposition 6.4. In particular, we show that in case of a positive friction coefficient α, equilibria correspond to the situation where the fluid is at rest.

One of the main results of this paper is contained in Theorem 6.6. It shows that in case $\dim \mathsf{M} = 2$, any solution with initial value $u_0 \in L_{2,\sigma}(\mathsf{M}; T\mathsf{M})$ exists globally and converges to an equilibrium at an exponential rate. Moreover, in case $\dim \mathsf{M} > 2$, we show in Theorem 6.7 and Corollary 6.9 that all equilibria are locally stable: solutions that start out close to an equilibrium exist globally and converge at an exponential rate to a (possibly different) equilibrium.

We add three examples to illustrate the scope of our results for two-dimensional surfaces in $\mathbb{R}^3$. For $\alpha \geq 0$, let $\mathcal{E}_\alpha$ denote the set of equilibria of (1.1).

Examples:

(a) Let $\mathsf{M} = \mathbb{S}^2_+ = \{x = (x_1, x_2, x_3) \in \mathbb{S}^2 : x_3 > 0\}$ be the upper hemisphere in $\mathbb{R}^3$. Then

$$\mathcal{E}_\alpha = \begin{cases} \{0\} & \text{in case } \alpha > 0, \\ \{\omega e_3 \times x : \omega \in \mathbb{R},\ x \in \mathbb{S}^2_+\} & \text{in case } \alpha = 0. \end{cases}$$

That is, in case of pure slip boundary conditions, the equilibria correspond to the situation where the fluid rotates with constant angular speed ω about the z-axis.
Theorem 6.6 says that in case $\alpha > 0$, any solution with initial value $u_0 \in L_{2,\sigma}(\mathsf{M}; T\mathsf{M})$ exists globally and converges at an exponential rate to the equilibrium state $u_\infty = 0$.
If $\alpha = 0$, any solution with initial value $u_0 \in L_{2,\sigma}(\mathsf{M}; T\mathsf{M})$ exists globally and converges at an exponential rate to an equilibrium state $u_\infty = \omega e_3 \times x$, for some $\omega \in \mathbb{R}$ which is determined by Theorem 6.6.

(b) Analogous results hold in case M is a disk in $\mathbb{R}^2$ with center at the origin (embedded in $\mathbb{R}^3$).

(c) Let $\mathsf{M} = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1, 0 < x_3 < 1\}$ be a cylinder of finite height. Then analogous results to Example (a) hold.

It is interesting to note that even in the simple Euclidean setting of Example (b), the results seem to be new, at least in the case where $\alpha = 0$.

For surfaces, in case $\alpha > 0$, the global convergence results are based on the fact that all Killing fields are trivial, see Proposition 6.2. In case $\alpha = 0$, the results follow from the somewhat surprising observation that the evolution equation leaves the orthogonal space to Killing fields invariant, see Lemma 6.5 (a), and from Korn's inequality, see Lemma B.3.

As another application of Theorem 6.7 we consider the three-dimensional manifold M consisting of a solid ball in $\mathbb{R}^3$ with center at the origin. Theorem 6.7 and Corollary 6.9 then show that rotations about any axis through the origin are stable: solutions that start close to a rotation exist globally and converge to a (possibly different) rotation. We are not aware of a corresponding result in the literature.

In the Appendices A through D we collect and prove results concerning Riemannian manifolds (with boundary), Green's formula, Korn's inequality, solvability of elliptic problems and the existence of the Helmholtz projection, interpolation for mixed boundary conditions, sectorial operators and the H^∞-calculus. These results are used throughout the manuscript and are also of independent interest.

In case M is an embedded hypersurface in $\mathbb{R}^{n+1}$ without boundary, the motion of an incompressible fluid has been considered in the literature by several authors. Here we refer to the article [6] for a survey and a comprehensive list of references. We also mention that the equations in $(1.1)_1$ and $(1.1)_2$ coincide with the system

$$\begin{cases} \partial_t u + \mathcal{P}_\mathsf{M}(u \cdot \nabla_\mathsf{M} u) - \mathcal{P}_\mathsf{M} \mathrm{div}_\mathsf{M}(2\mu_s \mathcal{D}_\mathsf{M}(u) - \pi \mathcal{P}_\mathsf{M}) = 0 & \text{on} \quad \mathsf{M}, \\ \mathrm{div}_\mathsf{M} u = 0 & \text{on} \quad \mathsf{M}, \end{cases}$$

considered in [30,37], see for instance [30, Remarks A.3]. In addition, we mention the publications [19,25,33,34] and the references contained therein for interesting numerical investigations for embedded surfaces in $\mathbb{R}^3$ without boundary.

For the case of a Riemannian manifold with boundary, we are aware of the publications [23,27]. The author in [27] considers Navier boundary conditions, and he examines the equations in a variational framework, mostly concentrating on the stationary linear case. In [23], the authors show that the Hodge-Laplacian subject to Neumann-type boundary conditions on a Lipschitz subdomain of a smooth, compact, boundaryless Riemannian manifold generates an analytic semigroup on L_q for q in some open interval containing $(3/2, 3)$.

In case of a domain contained in Euclidean space, the Navier-Stokes equations with Navier boundary conditions have been considered by numerous authors, and we refer to [32] for a discussion.

The novelty of this manuscript lies in the fact that we consider the behavior of fluids on surfaces, or manifolds, with boundaries. This situation extends traditional fluid dynamics analysis, which typically focuses on the Euclidean space. By studying fluids on manifolds, we are addressing a more complex scenario that also has applications.

For instance, this is relevant when analyzing the motion of water on a planet that is covered by both oceans and continents. In such a context, the surface of the planet can be modeled as a manifold with boundaries, representing land and sea.

In this situation, the analysis becomes considerably more complex than in the Euclidean case. Unlike in flat space, one must account for the manifold's geometric properties, which introduce additional mathematical challenges. Specifically, we need to handle geometric quantities such as the Ricci curvature, which incorporates how the manifold's shape deviates from being flat. These geometric considerations play a role for describing the behavior of fluids on curved surfaces.

When dealing with an impermeable boundary Σ, the most widely employed boundary condition in the literature is the no-slip condition, expressed as

$$u = 0 \quad \text{on} \quad \Sigma. \tag{1.4}$$

In contrast, the Navier boundary condition $(1.1)_3$ and $(1.1)_4$ permits tangential slip along the boundary. Over recent decades, a growing debate has emerged concerning the choice between the no-slip condition and the Navier condition, primarily due to the so-called no-collision paradox. Consider a rigid body in free fall within a fluid bounded by a solid wall. In case the rigid body and the wall have a smooth boundary, previous research [12,16,17] has demonstrated that under the assumption (1.4), the rigid body does not reach the fluid-solid interface in finite time, regardless of the relative densities of the fluid and the object. In contrast, assuming a Navier boundary condition circumvents such a situation [13].

Although we would expect similar results for the no-slip boundary conditions (1.4) as for the case of partial slip with $\alpha > 0$, the approach used here does not cover (1.4).

Notation. Given $q \in (1, \infty)$, $q' = q/(q-1)$ always denotes the Hölder conjugate of q.

Let X and Y be two Banach spaces and $T : X \to Y$. We denote by $\mathsf{D}(T)$, $\mathsf{N}(T)$ and $\mathsf{R}(T)$ the domain, null space and range of T, respectively. The notation $\mathcal{L}(X, Y)$ stands for the set of all bounded linear operators from X to Y and $\mathcal{L}(X) := \mathcal{L}(X, X)$. $\mathcal{L}is(X, Y)$ denotes the subset of $\mathcal{L}(X, Y)$ consisting of linear isomorphisms from X to Y. Moreover, we denote by $X' = \mathcal{L}(X, \mathbb{R})$ the dual of X.

For any $0 \le t_1 < t_2 < \infty$, $p \in (1, \infty)$ and $\mu \in (1/p, 1]$, the X-valued L_p-spaces with temporal weight are defined by

$$L_{p,\mu}((t_1,t_2);X) := \left\{ f : (t_1,t_2) \to X : t \mapsto t^{1-\mu} f(t) \in L_p((t_1,t_2);X) \right\}.$$

Similarly,

$$H^k_{p,\mu}((t_1,t_2);X) := \left\{ f \in W^k_{1,loc}((t_1,t_2);X) : \partial_t^j f \in L_{p,\mu}((t_1,t_2);X),\ j=0,1,\dots,k \right\}.$$

2. The surface Stokes operator with Navier boundary conditions

To analyze (1.1), we introduce the *surface Helmholtz projection*, defined by

$$\mathbb{P}_H u = u - \operatorname{grad}\psi_u, \quad u \in L_q(\mathsf{M};T\mathsf{M}),$$

where $\operatorname{grad}\psi_u \in L_q(\mathsf{M};T\mathsf{M})$ is the unique solution of

$$(\operatorname{grad}\psi_u|\operatorname{grad}\phi)_{\mathsf{M}} = (u|\operatorname{grad}\phi)_{\mathsf{M}}, \quad \forall \phi \in \dot{H}^1_{q'}(\mathsf{M}),$$

cf. Lemma B.6. Here,

$$(u|v)_{\mathsf{M}} := \int_{\mathsf{M}} (u|v)_g \, d\mu_g, \quad (u,v) \in L_r(\mathsf{M};T\mathsf{M}) \times L_{r'}(\mathsf{M};T\mathsf{M}),$$

denotes the duality pairing between $L_q(\mathsf{M};T\mathsf{M})$ and $L_{q'}(\mathsf{M};T\mathsf{M})$. We note that in case $q=2$, the pairing $(\cdot|\cdot)_{\mathsf{M}}$ defines an inner product on $L_2(\mathsf{M};T\mathsf{M})$.

For any $u \in L_q(\mathsf{M};T\mathsf{M})$ and $v \in L_{q'}(\mathsf{M};T\mathsf{M})$ it holds

$$\begin{aligned}(\mathbb{P}_H u|v)_{\mathsf{M}} &= (u - \operatorname{grad}\psi_u|v)_{\mathsf{M}} = (u|v)_{\mathsf{M}} - (\operatorname{grad}\psi_u|v)_{\mathsf{M}} \\ &= (u|v)_{\mathsf{M}} - (\operatorname{grad}\psi_u|\operatorname{grad}\psi_v)_{\mathsf{M}} = (u|v)_{\mathsf{M}} - (u|\operatorname{grad}\psi_v)_{\mathsf{M}} \\ &= (u|\mathbb{P}_H v)_{\mathsf{M}}\end{aligned} \tag{2.1}$$

as $\psi_u \in \dot{H}^1_q(\mathsf{M})$ and $\psi_v \in \dot{H}^1_{q'}(\mathsf{M})$. Note that the definition of $\mathbb{P}_H$ implies

$$(u|\nu_\Sigma)_g = 0 \text{ on } \Sigma \ \text{ in case } \ u \in H^s_{q,\sigma}(\mathsf{M};T\mathsf{M}) \text{ and } s > 1/q. \tag{2.2}$$

With these preparations, we can introduce the function spaces used in this article

$$\begin{aligned} L_{q,\sigma}(\mathsf{M};T\mathsf{M}) &:= \mathbb{P}_H L_q(\mathsf{M};T\mathsf{M}) \\ H^s_{q,\sigma}(\mathsf{M};T\mathsf{M}) &:= H^s_q(\mathsf{M};T\mathsf{M}) \cap L_{q,\sigma}(\mathsf{M};T\mathsf{M}) \\ B^s_{qp,\sigma}(\mathsf{M};T\mathsf{M}) &:= B^s_{qp}(\mathsf{M};T\mathsf{M}) \cap L_{q,\sigma}(\mathsf{M};T\mathsf{M}) \\ H^{-s}_{q,\sigma}(\mathsf{M};T\mathsf{M}) &:= (H^s_{q',\sigma}(\mathsf{M};T\mathsf{M}))' \\ B^{-s}_{qp,\sigma}(\mathsf{M};T\mathsf{M}) &:= (B^s_{q'p',\sigma}(\mathsf{M};T\mathsf{M}))' \end{aligned} \tag{2.3}$$

for $s \geq 0$ and $1 < p, q < \infty$, where the respective duality parings

$$\langle\cdot|\cdot\rangle_{\mathsf{M}} : H^{-s}_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \times H^{s}_{q',\sigma}(\mathsf{M}; T\mathsf{M})) \to \mathbb{R},$$
$$\langle\cdot|\cdot\rangle_{\mathsf{M}} : B^{-s}_{qp,\sigma}(\mathsf{M}; T\mathsf{M}) \times B^{s}_{q'p',\sigma}(\mathsf{M}; T\mathsf{M}) \to \mathbb{R},$$

are induced by $(\cdot|\cdot)_{\mathsf{M}}$. We would like to point out that our definition of the 'negative' spaces H^{-s}_q and B^{-s}_{qp} differs from the usual definition in case $-s < -1/q'$. This allows for a more streamlined presentation of our results. As the spaces involved will be clear from the context they will not be explicitly referenced in our notation $\langle\cdot|\cdot\rangle_{\mathsf{M}}$. Note that

$$\langle u|v\rangle_{\mathsf{M}} = (u|v)_{\mathsf{M}} \quad \text{in case} \quad (u, v) \in L_q(\mathsf{M}; T\mathsf{M}) \times L_{q'}(\mathsf{M}; T\mathsf{M}).$$

Now we can define the *strong surface Stokes operator with Navier boundary conditions*, $A_N : X_1 \to X_0$, by

$$A_N u := -2\mu_s \mathbb{P}_H \operatorname{div} D(u) = -\mu_s \mathbb{P}_H(\Delta_{\mathsf{M}} u + \mathsf{Ric}^\sharp u) \tag{2.4}$$

with $X_0 := L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$ and

$$X_1 := \mathsf{D}(A_N) := \{u \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}) : (u|\nu_\Sigma)_g = 0,\ \ \alpha u + \mathcal{P}_\Sigma\big((\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma\big) = 0 \text{ on } \Sigma\}. \tag{2.5}$$

Although the condition $(u|\nu_\Sigma)_g = 0$ is already contained in the stipulation $u \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})$, see (2.2), we include it in the definition for extra emphasis.

Next, we will derive a simpler expression of the boundary conditions of (1.1). We first note that in local coordinates

$$\nu_\Sigma = \sum_{j=1}^n \frac{1}{\sqrt{g^{nn}}} g^{nj} \frac{\partial}{\partial x^j} = \frac{1}{\sqrt{g^{nn}}} g^{nj} \frac{\partial}{\partial x^j}. \tag{2.6}$$

In addition, we set $\mathcal{P}_\Sigma = I_{T\mathsf{M}} - \dfrac{1}{g^{nn}} g^{nj} \dfrac{\partial}{\partial x^j} \otimes dx^n$. Hence,

$$\mathcal{P}_\Sigma \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, n-1, \qquad \mathcal{P}_\Sigma\left(g^{nj}\frac{\partial}{\partial x^j}\right) = 0. \tag{2.7}$$

Then we have for any $u \in H^1_{q,\sigma}(\mathsf{M}; T\mathsf{M})$, using the metric property of $(\cdot|\cdot)_g$, (2.7), the boundary condition $(u|\nu_\Sigma)_g = 0$, and (A.1)

$$\begin{aligned}
\mathcal{P}_\Sigma\left([\nabla u]^{\mathsf{T}}\nu_\Sigma\right) &= \mathcal{P}_\Sigma\left(g^\sharp(dx^i \otimes \nabla_i u)g_\flat \nu_\Sigma\right) = \mathcal{P}_\Sigma\left(g^\sharp dx^i (\nabla_i u|\nu_\Sigma)_g\right) \\
&= \sum_{i,j=1}^n \left[\nabla_i (u|\nu_\Sigma)_g - (u|\nabla_i \nu_\Sigma)_g\right] \mathcal{P}_\Sigma\, g^{ij} \frac{\partial}{\partial x^j} \\
&= \sum_{i=1}^{n-1}\sum_{j=1}^n \left[\nabla_i (u|\nu_\Sigma)_g - (u|\nabla_i \nu_\Sigma)_g\right] \mathcal{P}_\Sigma\, g^{ij} \frac{\partial}{\partial x^j}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1}\sum_{j=1}^{n}(L_\Sigma u|\frac{\partial}{\partial x^i})_g\,\mathcal{P}_\Sigma\, g^{ij}\frac{\partial}{\partial x^j} \\
&= \sum_{i,j=1}^{n}(L_\Sigma u|\frac{\partial}{\partial x^i})_g\,\mathcal{P}_\Sigma\, g^{ij}\frac{\partial}{\partial x^j} = \sum_{i,j=1}^{n}(L_\Sigma u|g^{ij}\frac{\partial}{\partial x^i})_g\,\mathcal{P}_\Sigma\frac{\partial}{\partial x^j} \\
&= \sum_{j=1}^{n}\langle dx^j, L_\Sigma u\rangle_{g_\Sigma}\mathcal{P}_\Sigma\frac{\partial}{\partial x^j} = \sum_{j=1}^{n-1}\langle dx^j, L_\Sigma u\rangle_{g_\Sigma}\frac{\partial}{\partial x^j} = L_\Sigma u,
\end{aligned} \tag{2.8}
$$

where L_Σ is the Weingarten tensor induced by $g|_\Sigma$, with $L_\Sigma u \in \Gamma(\Sigma, T\Sigma)$, and

$$\langle\cdot,\cdot\rangle_{g_\Sigma} : T^*\Sigma \times T\Sigma \to \mathbb{R}^\Sigma \quad \text{is the (fiber-wise defined) duality pairing on } \Sigma.$$

Moreover,

$$
\begin{aligned}
\mathcal{P}_\Sigma(\nabla u \nu_\Sigma) &= \frac{1}{\sqrt{g^{nn}}}g^{nj}\mathcal{P}_\Sigma\left(\nabla_j u\right) = \frac{1}{\sqrt{g^{nn}}}g^{nj}\mathcal{P}_\Sigma\left((\partial_j u^i + \Gamma^i_{kj}u^k)\frac{\partial}{\partial x^i}\right) \\
&= \frac{g^{nj}}{\sqrt{g^{nn}}}(\partial_j u^i + \Gamma^i_{kj}u^k)\frac{\partial}{\partial x^i} - \frac{g^{nj}g^{ni}}{(g^{nn})^{3/2}}(\partial_j u^n + \Gamma^n_{kj}u^k)\frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^{n-1}\left[\frac{g^{nj}}{\sqrt{g^{nn}}}(\partial_j u^i + \Gamma^i_{kj}u^k) - \frac{g^{nj}g^{ni}}{(g^{nn})^{3/2}}(\partial_j u^n + \Gamma^n_{kj}u^k)\right]\frac{\partial}{\partial x^i}.
\end{aligned} \tag{2.9}
$$

By (2.1), (2.8), and Lemma B.1 (b) (ii), for any $u \in \mathsf{D}(A_N)$ and $v \in H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})$,

$$
\begin{aligned}
(A_N u|v)_{\mathsf{M}} &= \mu_s(\nabla u|\nabla v)_{\mathsf{M}} - \mu_s(\mathsf{Ric}^\sharp u|v)_{\mathsf{M}} - \mu_s(\nabla u\nu_\Sigma|v)_\Sigma \\
&= \mu_s(\nabla u|\nabla v)_{\mathsf{M}} - \mu_s(\mathsf{Ric}^\sharp u|v)_{\mathsf{M}} - \mu_s(\mathcal{P}_\Sigma(\nabla u)\nu_\Sigma|v)_\Sigma \\
&= \mu_s(\nabla u|\nabla v)_{\mathsf{M}} - \mu_s(\mathsf{Ric}^\sharp u|v)_{\mathsf{M}} + (\alpha\mu_s u + \mu_s L_\Sigma u|v)_\Sigma,
\end{aligned} \tag{2.10}
$$

where $(\cdot|\cdot)_\Sigma$ denotes the duality pairing between $L_q(\Sigma, T\Sigma)$ and $L_{q'}(\Sigma; T\Sigma)$. By setting

$$X_{1/2} := H^1_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \quad \text{and} \quad X_{-1/2} := \left(H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})\right)' =: H^{-1}_{q,\sigma}(\mathsf{M}; T\mathsf{M}), \tag{2.11}$$

the above computations motivate us to define the *weak surface Stokes operator with Navier boundary conditions* $A^{\mathsf{w}}_N : X_{1/2} \to X_{-1/2}$ by

$$\langle A^{\mathsf{w}}_N u|v\rangle_{\mathsf{M}} = \mu_s(\nabla u|\nabla v)_{\mathsf{M}} - \mu_s(\mathsf{Ric}^\sharp u|v)_{\mathsf{M}} + (\alpha\mu_s u + \mu_s L_\Sigma u|v)_\Sigma$$

for all $(u, v) \in X_{1/2} \times (X_{-1/2})' = H^1_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \times H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})$.

The next result states that the surface Stokes operators A^{w}_N and A_N both admit a bounded H^∞-calculus.

Theorem 2.1. *There exists a number $\omega_0 > 0$ such that*

(a) $\omega + A_N^{\mathsf{W}} \in H^\infty(X_{-1/2})$ *with* H^∞*-angle* $< \pi/2$ *for all* $\omega > \omega_0$.
(b) $\omega + A_N \in H^\infty(X_0)$ *with* H^∞*-angle* $< \pi/2$ *for all* $\omega > \omega_0$.

Proof. For a proof we refer to Section 4. □

3. H^∞-calculus of surface Stokes operator with perfect slip boundary conditions

In order to prove Theorem 2.1, we take a detour and first consider the Stokes operator with perfect slip boundary conditions, which is also of interest in its own right. On a technical level, we are aided by the fact that in this case, the Helmholtz projection commutes with the Laplacian, which (temporarily) allows us to ignore the pressure and the divergence condition. The same strategy was also employed in [32].

We start this section by providing the necessary tools to localize differential equations and tangent fields that are defined on the manifold M. We define an atlas $\{(U_k, \varphi_k)\}_{k\in\mathcal{K}}$ of M with $\mathcal{K} = \mathcal{K}_0 \sqcup \mathcal{K}_1$ such that $k \in \mathcal{K}_0$ if $U_k \cap \Sigma = \emptyset$ and $k \in \mathcal{K}_1$ if $U_k \cap \Sigma \neq \emptyset$. Moreover,

$$\varphi_k(U_k) = \mathbb{B}^n_k(0, R) := \begin{cases} \mathbb{B}^n(0, R), & \text{if } k \in \mathcal{K}_0, \\ \mathbb{B}^n(0, R) \cap \mathbb{R}^n_+, & \text{if } k \in \mathcal{K}_1. \end{cases}$$

Given $k \in \mathcal{K}$, we set

$$\mathbb{X}_k = \begin{cases} \mathbb{R}^n, & \text{if } k \in \mathcal{K}_0, \\ \mathbb{R}^n_+, & \text{if } k \in \mathcal{K}_1, \end{cases}$$

endowed with the Euclidean metric in $\mathbb{R}^n$. Let $\{\xi_k^2\}_{k\in\mathcal{K}}$ be a partition of unity subject to $\{U_k\}_{k\in\mathcal{K}}$. Furthermore, let $\zeta \in C_0^\infty(\mathbb{B}^n(0, R); [0, 1])$ be chosen such that

$$\zeta \equiv 1 \quad \text{on } \operatorname{supp}((\varphi_k)_* \xi_k) \quad \text{for all } k \in \mathcal{K},$$

where $(\varphi_k)_*\phi := \phi \circ \varphi_k^{-1}$ is the pushforward of a function $\phi : \mathsf{M} \to \mathbb{R}$ by φ_k. Given $u \in \Gamma(\mathsf{M}; T\mathsf{M})$, we define

$$(\varphi_k)_* u := ((\varphi_k)_* u^i)_{1\le i\le n}, \qquad \text{where } u = u^i \frac{\partial}{\partial x^i}.$$

For $\mathfrak{F} \in \{H_q, W_q\}$, $1 < q < \infty$, and $s \ge 0$, we define:

$$\begin{aligned} &\mathcal{R}_k^c : \mathfrak{F}^s(\mathsf{M}; T\mathsf{M}) \to \mathfrak{F}^s(\mathbb{X}_k; \mathbb{R}^n), && u \mapsto (\varphi_k)_*(\xi_k u), \\ &\mathcal{R}_k : \mathfrak{F}^s(\mathbb{X}_k; \mathbb{R}^n) \to \mathfrak{F}^s(\mathsf{M}; T\mathsf{M}), && u_\kappa \mapsto \xi_k(\varphi_k^* u_\kappa). \end{aligned}$$

Here and in the following, it is understood that a partially defined and compactly supported vector field is automatically extended over the whole base manifold by identifying it to be the zero section outside its original domain.

With a slight abuse of notation, we define the pullback of a vector field $v : \mathbb{X}_k \to \mathbb{R}^n$ by means of

$$\varphi_k^* v := \left(v^i \circ \varphi_k\right) \frac{\partial}{\partial x^i}.$$

Finally, we define

$$\begin{aligned}
&\mathcal{R}^c : \mathfrak{F}^s(\mathsf{M}; T\mathsf{M}) \to \boldsymbol{\mathfrak{F}}^s(\mathbb{X}; \mathbb{R}^n), \quad u \mapsto (\mathcal{R}_k^c u)_{k\in\mathcal{K}},\\
&\mathcal{R} :\ \boldsymbol{\mathfrak{F}}^s(\mathbb{X}; \mathbb{R}^n) \to \mathfrak{F}^s(\mathsf{M}; T\mathsf{M}), \quad \boldsymbol{v} = (v_k)_{k\in\mathcal{K}} \mapsto \sum_k \mathcal{R}_k v_k
\end{aligned}$$

with $\boldsymbol{\mathfrak{F}}^s(\mathbb{X}; \mathbb{R}^n) := \prod_{k\in\mathcal{K}} \mathfrak{F}^s(\mathbb{X}_k; \mathbb{R}^n)$, equipped with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{\mathfrak{F}}^s} = \sum_{k\in\mathcal{K}} \|v_k\|_{\mathfrak{F}^s(\mathbb{X}_k)}, \quad \boldsymbol{v} = (v_k)_{k\in\mathcal{K}}.$$

Then one shows that

$$\mathcal{R}^c \in \mathcal{L}(\mathfrak{F}^s(\mathsf{M}; T\mathsf{M}), \boldsymbol{\mathfrak{F}}^s(\mathbb{X}; \mathbb{R}^n)), \quad \mathcal{R} \in \mathcal{L}(\boldsymbol{\mathfrak{F}}^s(\mathbb{X}; \mathbb{R}^n), \mathfrak{F}^s(\mathsf{M}; T\mathsf{M})),$$

see for instance [4]. Moreover,

$$(\mathcal{R} \circ \mathcal{R}^c)u = u, \quad u \in \mathfrak{F}^s(\mathsf{M}; T\mathsf{M}),$$

that is, $\mathcal{R}$ is a retraction from $\boldsymbol{\mathfrak{F}}^s(\mathbb{X}; \mathbb{R}^n)$ onto $\mathfrak{F}^s(\mathsf{M}; T\mathsf{M})$, and $\mathcal{R}^c$ is a coretraction.

3.1. *Strong formulation*

Following the ideas of [32], we will first study the Stokes operator with *perfect slip boundary conditions*. To this end, we consider first the elliptic boundary value problem,

$$\begin{cases}
(\lambda - \Delta_{\mathsf{M}} + \mathsf{Ric}^\sharp)u = f & \text{on} \quad \mathsf{M},\\
\mathcal{P}_\Sigma\left((\nabla u - [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) = h_1 & \text{on} \quad \Sigma,\\
(u|\nu_\Sigma)_g = h_2 & \text{on} \quad \Sigma,
\end{cases} \tag{3.1}$$

for suitable $\lambda \in \mathbb{C}$ and

$$(f, h_1, h_2) \in L_q(\mathsf{M}; T\mathsf{M}) \times W_q^{1-1/q}(\Sigma; T\Sigma) \times W_q^{2-1/q}(\Sigma).$$

We should like to briefly explain our rationale for using the terminology *perfect slip boundary conditions*. In three-dimensional Euclidean space, it can be shown that

$$\mathcal{P}_\Sigma\left((\nabla u - [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) = \nu_\Sigma \times \operatorname{curl} u,$$

see for instance [29, Section 4.1]. In applications in (magneto) hydrodynamics, the boundary conditions

$$(u|\nu_\Sigma)_g = 0, \quad \operatorname{curl} u \times \nu_\Sigma = 0,$$

are sometimes referred to as *perfect wall conditions,* see for instance [1]. In addition, these conditions are also known as *Neumann boundary conditions* or *free boundary conditions,* see for instance [23,24] and the reference therein. For lack of a better name and following [29,32], we will use the same terminology also in the general situation of manifolds of arbitrary dimension.

We have the following result about existence and uniqueness of solutions to (3.1).

Proposition 3.1. *Let $1 < q < \infty$ and $\phi \in (0, \pi/2)$. Then, there exists a number $\lambda_0 > 0$ such that for all $\lambda \in \lambda_0 + \Sigma_{\pi-\phi}$ problem (3.1) has a unique solution $u \in H^2_q(\mathsf{M}; T\mathsf{M})$ if and only if*

$$(f, h_1, h_2) \in L_q(\mathsf{M}; T\mathsf{M}) \times W^{1-1/q}_q(\Sigma; T\Sigma) \times W^{2-1/q}_q(\Sigma).$$

Furthermore, there exists a constant $C > 0$ such that for all $\lambda \in \lambda_0 + \Sigma_{\pi-\phi}$ the estimate

$$|\lambda|\|u\|_{L_q(\mathsf{M})} + \|u\|_{H^2_q(\mathsf{M})} \le C\Big(\|f\|_{L_q(\mathsf{M})} + \|H_1\|_{H^1_q(\mathsf{M})+|\lambda|^{1/2}\|H_1\|_{L_q(\mathsf{M})}} + \|H_2\|_{H^2_q(\mathsf{M})} + |\lambda|^{1/2}\|H_2\|_{H^1_q(\mathsf{M})} + |\lambda|\|H_2\|_{L_q(\mathsf{M})}\Big) \tag{3.2}$$

holds, where H_j is any extension of h_j from $W^{j-1/q}_q(\Sigma)$ to $H^j_q(\mathsf{M})$.

Proof. In short form, (3.1) can be formulated as

$$L_\lambda u = F, \tag{3.3}$$

where $L_\lambda : H^2_q(\mathsf{M}; T\mathsf{M}) \to L_q(\mathsf{M}; T\mathsf{M}) \times W^{1-1/q}_q(\Sigma; T\Sigma) \times W^{2-1/q}_q(\Sigma)$ is defined by the left side of (3.1) and $F := (f, h_1, h_2)$.

In the following, we will show that the operator L_λ is invertible for λ appropriately chosen. We start by establishing a priori estimates for solutions of (3.1). Suppose $u \in H^2_q(\mathsf{M}; T\mathsf{M})$ is a solution of (3.1). We then set

$$\bar{u}_k := \mathcal{R}^c_k u = (\bar{u}^1_k, \bar{u}^2_k, \cdots, \bar{u}^n_k)^\mathsf{T}$$

and

$$\bar{G}_{(k)} = [\bar{g}^{ij}_{(k)}]^i_j = \zeta G_k + (1-\zeta) I_n,$$

where $G_k := [(\varphi_k)_* g^{ij}]^i_j$. Using these notations and (1.3), we can write the first line in (3.1) in local coordinates as

$$(\lambda - \bar{g}^{ij}_{(k)} \partial_i \partial_j)\bar{u}_k = \bar{f}_k + P_k(u) \quad \text{in } \mathbb{X}_k, \tag{3.4}$$

where the matrices $\bar{G}_{(k)}$ belong to $BC^\infty(\mathbb{X}_k; \mathbb{R}^{n\times n})$ and $\bar{f}_k := \mathcal{R}^c_k f$. Up to translations and rotations, $\|\bar{G}_{(k)} - I_n\|_\infty$ can be made arbitrarily small by shrinking the radius $R > 0$ of $\mathbb{B}^n_k(0, R)$. The linear operator P_k is of first order; in particular

$$P_k \in \mathcal{L}(H^1_q(\mathsf{M}; T\mathsf{M}), L_q(\mathbb{X}_k; \mathbb{R}^n)).$$

Next, we will localize the boundary conditions in (3.1). First, in view of (2.6), the boundary condition $(u|\nu_\Sigma)_g = h_2$ can be restated as

$$\frac{1}{\sqrt{\bar{g}^{nn}_{(k)}}}\bar{u}^n_k = \bar{h}_{2,k} \quad \text{on } \mathbb{R}^{n-1}, \quad k \in \mathcal{K}_1. \tag{3.5}$$

Using (2.8) and (2.9), the remaining boundary condition $\mathcal{P}_\Sigma\big((\nabla u - [\nabla u]^\mathsf{T})\nu_\Sigma\big) = h_1$ can be rewritten as

$$\frac{1}{\sqrt{\bar{g}^{nn}_{(k)}}}\bar{g}^{nj}_{(k)}\partial_j \bar{u}^i_k - \frac{\bar{g}^{nj}_{(k)}\bar{g}^{ni}_{(k)}}{(\bar{g}^{nn}_{(k)})^{3/2}}\partial_j \bar{u}^n_k = \bar{h}^i_{1,k} + \mathrm{tr}_{\mathbb{R}^{n-1}} Q^i_k(u) \quad \text{on } \mathbb{R}^{n-1}, \tag{3.6}$$

for $k \in \mathcal{K}_1$, $i = 1, 2, \dots, n-1$. We note here that $Q^i_k(u)$, in particular, contain an extension of the (localized) term $L_\Sigma u$ in (2.8) to $H^2_q(\mathbb{X}_k)$. It follows that $Q^i_k \in \mathcal{L}(H^2_q(\mathsf{M}; T\mathsf{M}), H^2_q(\mathbb{X}_k))$ with

$$\|Q^i_k(u)\|_{H^s_q(\mathbb{X}_k)} \le C(s)\|u\|_{H^s_q(\mathsf{M};T\mathsf{M})}, \quad u \in H^2_q(\mathsf{M}; T\mathsf{M}),$$

for any $s \in [0, 2]$. We define

$$L^\#_{\lambda,k} : H^2_q(\mathbb{X}_k; \mathbb{R}^n) \to L_q(\mathbb{X}_k; \mathbb{R}^n)$$

for $k \in \mathcal{K}_0$ by $L^\#_{\lambda,k} v := \lambda v - \bar{g}^{ij}_{(k)}\partial_i\partial_j v$ and

$$L^\#_{\lambda,k} : H^2_q(\mathbb{X}_k; \mathbb{R}^n) \to L_q(\mathbb{X}_k; \mathbb{R}^n) \times W^{1-1/q}_q(\mathbb{R}^{n-1}; \mathbb{R}^{n-1}) \times W^{2-1/q}_q(\mathbb{R}^{n-1})$$

for $k \in \mathcal{K}_1$ by

$$L^\#_{\lambda,k} v := \begin{pmatrix} \lambda v - \bar{g}^{ij}_{(k)}\partial_i\partial_j v \\ \mathrm{tr}_{\mathbb{R}^{n-1}}\big(\frac{1}{\sqrt{\bar{g}^{nn}_{(k)}}}\bar{g}^{nj}_{(k)}\partial_j v^1 - \frac{\bar{g}^{nj}_{(k)}\bar{g}^{n1}_{(k)}}{(\bar{g}^{nn}_{(k)})^{3/2}}\partial_j v^n\big) \\ \vdots \\ \mathrm{tr}_{\mathbb{R}^{n-1}}\big(\frac{1}{\sqrt{\bar{g}^{nn}_{(k)}}}\bar{g}^{nj}_{(k)}\partial_j v^{n-1} - \frac{\bar{g}^{nj}_{(k)}\bar{g}^{n,n-1}_{(k)}}{(\bar{g}^{nn}_{(k)})^{3/2}}\partial_j v^n\big) \\ \mathrm{tr}_{\mathbb{R}^{n-1}}\frac{1}{\sqrt{\bar{g}^{nn}_{(k)}}} v^n \end{pmatrix},$$

where $\mathrm{tr}_{\mathbb{R}^{n-1}}$ is the trace operator from $\mathbb{R}^n_+$ to $\mathbb{R}^{n-1}$.

Let us denote by $L^{\#,0}_{\lambda,k}$ the corresponding operator in the planar case, i.e. $G_k = I_n$. Then it holds that $L^{\#,0}_{\lambda,k} v = \lambda v - \Delta_{\mathbb{R}^n} v$ if $k \in \mathcal{K}_0$ and

$$L^{\#,0}_{\lambda,k} v = \Big(\lambda v - \Delta_{\mathbb{R}^n_+} v, \mathrm{tr}_{\mathbb{R}^{n-1}}\partial_n v^1, \cdots, \mathrm{tr}_{\mathbb{R}^{n-1}}\partial_n v^{n-1}, \mathrm{tr}_{\mathbb{R}^{n-1}} v^n\Big)^\mathsf{T}$$

if $k \in \mathcal{K}_1$. It is well-known that for each $\lambda \in \Sigma_{\pi-\phi}$, $\phi \in (0, \pi/2)$, the operators $L^{\#,0}_{\lambda,k}$ are isomorphisms between the corresponding spaces defined above. It follows from [22, Theorem 3.1.3] that there exists a constant $C > 0$ being independent of λ, such that the unique solution v to the elliptic problems

$$L^{\#,0}_{\lambda,k} v = \bar{f}$$

for $k \in \mathcal{K}_0$, $\bar{f} \in L_q(\mathbb{R}^n; \mathbb{R}^n)$ and

$$L^{\#,0}_{\lambda,k} v = (\bar{f}, \bar{h}_1^1, \ldots, \bar{h}_1^{n-1}, \bar{h}_2)^{\mathsf{T}}$$

for $k \in \mathcal{K}_1$ and

$$(\bar{f}, \bar{h}_1, \bar{h}_2) \in L_q(\mathbb{R}^n_+; \mathbb{R}^n) \times W_q^{1-1/q}(\mathbb{R}^{n-1}; \mathbb{R}^{n-1}) \times W_q^{2-1/q}(\mathbb{R}^{n-1})$$

satisfies

$$\begin{aligned} |\lambda| \|v\|_{L_q(\mathbb{X}_k)} + \|v\|_{H^2_q(\mathbb{X}_k)} \leq C\Big(&\|\bar{f}\|_{L_q(\mathbb{X}_k)} + \|\bar{H}_1\|_{H^1_q(\mathbb{X}_k)} + |\lambda|^{1/2} \|\bar{H}_1\|_{L_q(\mathbb{X}_k)} \\ &+ \|\bar{H}_2\|_{H^2_q(\mathbb{X}_k)} + |\lambda|^{1/2} \|\bar{H}_2\|_{H^1_q(\mathbb{X}_k)} + |\lambda| \|\bar{H}_2\|_{L_q(\mathbb{X}_k)} \Big) \end{aligned} \tag{3.7}$$

for all $\lambda \in \Sigma_{\pi-\phi}$, $\phi \in (0, \pi/2)$ and any extension $\bar{H}_j$ of $\bar{h}_j$ from $W_q^{j-1/q}(\mathbb{R}^{n-1})$ to $H^j_q(\mathbb{X}_k)$. In case $k \in \mathcal{K}_0$, the terms in (3.7) containing $\bar{H}_1$ and $\bar{H}_2$ are omitted.

We will show in the sequel, that (3.7) still holds for the general geometry by means of a perturbation argument. To this end, we write

$$L^{\#}_{\lambda,k} = L^{\#,0}_{\lambda,k} + L^{\#}_{\lambda,k} - L^{\#,0}_{\lambda,k},$$

wherefore the equation $L^{\#}_{\lambda,k} v = (\bar{f}, \bar{h}_1^1, \ldots, \bar{h}_1^{n-1}, \bar{h}_2)^{\mathsf{T}}$ is equivalent to

$$v + [L^{\#,0}_{\lambda,k}]^{-1}(L^{\#}_{\lambda,k} - L^{\#,0}_{\lambda,k})v = [L^{\#,0}_{\lambda,k}]^{-1}(\bar{f}, \bar{h}_1^1, \ldots, \bar{h}_1^{n-1}, \bar{h}_2)^{\mathsf{T}}.$$

In case $k \in \mathcal{K}_0$, it holds that

$$L^{\#}_{\lambda,k} v - L^{\#,0}_{\lambda,k} v = \Delta_{\mathbb{R}^n} v - \bar{g}^{ij}_{(k)} \partial_i \partial_j v,$$

and

$$\begin{aligned} \|\Delta_{\mathbb{R}^n} v - \bar{g}^{ij}_{(k)} \partial_i \partial_j v\|_{L_q(\mathbb{X}_k)} &\leq \|\bar{G}_{(k)} - I_n\|_\infty \|v\|_{H^2_q(\mathbb{X}_k)} \\ &\leq \|\bar{G}_{(k)} - I_n\|_\infty \Big(|\lambda| \|v\|_{L_q(\mathbb{X}_k)} + \|v\|_{H^2_q(\mathbb{X}_k)} \Big), \end{aligned}$$

where we recall that $\|\bar{G}_{(k)} - I_n\|_\infty$ can be made as small as we wish. Therefore we may achieve

$$\|[L^{\#,0}_{\lambda,k}]^{-1}(L^{\#,0}_{\lambda,k} - L^{\#}_{\lambda,k})v\|_{H^2_{q,\lambda}(\mathbb{X}_k)} \leq \frac{1}{2} \|v\|_{H^2_{q,\lambda}(\mathbb{X}_k)},$$

where $H^2_{q,\lambda}(\mathbb{X}_k)$ denotes the space $H^2_q(\mathbb{X}_k)$ equipped with the norm $|\lambda|\,\|\cdot\|_{L_q(\mathbb{X}_k)} + \|\cdot\|_{H^2_q(\mathbb{X}_k)}$. A Neumann series argument then yields that the linear operator

$$I + [L^{\#,0}_{\lambda,k}]^{-1}(L^{\#}_{\lambda,k} - L^{\#,0}_{\lambda,k}) : H^2_{q,\lambda}(\mathbb{X}_k) \to H^2_{q,\lambda}(\mathbb{X}_k)$$

is invertible and the estimate

$$\|v\|_{H^2_{q,\lambda}(\mathbb{X}_k)} \le 2C\|\bar{f}\|_{L_q(\mathbb{X}_k)}$$

holds, whenever $k \in \mathcal{K}_0$.

If $k \in \mathcal{K}_1$, then perturbations on the boundary $\partial\mathbb{X}_k = \mathbb{R}^{n-1}$ have to be taken into account. We show this exemplarily for the last component of $L^{\#}_{\lambda,k}v - L^{\#,0}_{\lambda,k}v$, given by

$$\mathrm{tr}_{\mathbb{R}^{n-1}}((\bar{g}^{nn}_{(k)})^{-1/2}v^n - v^n) \in W^{2-1/q}_q(\mathbb{R}^{n-1}).$$

The term $((\bar{g}^{nn}_{(k)})^{-1/2} - 1)v^n \in H^2_q(\mathbb{X}_k)$ is an extension and will be estimated with respect to the norm

$$\|\cdot\|_{H^2_q(\mathbb{X}_k)} + |\lambda|^{1/2}\|\cdot\|_{H^1_q(\mathbb{X}_k)} + |\lambda|\|\cdot\|_{L_q(\mathbb{X}_k)}.$$

For the sake of readability, we write $a = (\bar{g}^{nn}_{(k)})^{-1/2} - 1$ and we recall that $\|a\|_{L_\infty(\mathbb{X}_k)}$ can be made as small as we wish, while $\|a\|_{W^2_\infty(\mathbb{X}_k)}$ is bounded. Then, it holds that

$$|\lambda|\|av^n\|_{L_q(\mathbb{X}_k)} \le |\lambda|\|a\|_{L_\infty(\mathbb{X}_k)}\|v^n\|_{L_q(\mathbb{X}_k)} \le |\lambda|\|a\|_{L_\infty(\mathbb{X}_k)}\|v\|_{L_q(\mathbb{X}_k)} \le \|a\|_{L_\infty(\mathbb{X}_k)}\|v\|_{H^2_{q,\lambda}(\mathbb{X}_k)}$$

by the definition of the norm in $H^2_{q,\lambda}(\mathbb{X}_k)$. Furthermore, we have

$$\|av^n\|_{H^1_q(\mathbb{X}_k)} \le \|a\|_{W^1_\infty(\mathbb{X}_k)}\|v\|_{L_q(\mathbb{X}_k)} + \|a\|_{L_\infty(\mathbb{X}_k)}\|v\|_{H^1_q(\mathbb{X}_k)},$$

where

$$\|v\|_{L_q(\mathbb{X}_k)} \le |\lambda|^{-1}\|v\|_{H^2_{q,\lambda}(\mathbb{X}_k)}.$$

We make use of complex interpolation

$$H^1_q(\mathbb{X}_k) = [L_q(\mathbb{X}_k), H^2_q(\mathbb{X}_k)]_{1/2}$$

and Young's inequality to obtain

$$\|v\|_{H^1_q(\mathbb{X}_k)} \le C|\lambda|^{-1/2}\|v\|_{H^2_{q,\lambda}(\mathbb{X}_k)}.$$

This then implies that

$$|\lambda|^{1/2}\|av^n\|_{H^1_q(\mathbb{X}_k)} \le C\left(\|a\|_{W^1_\infty(\mathbb{X}_k)}|\lambda|^{-1/2} + \|a\|_{L_\infty(\mathbb{X}_k)}\right)\|v\|_{H^2_{q,\lambda}(\mathbb{X}_k)}.$$

Finally, to estimate av^n in $W^2_q(\mathbb{X}_k)$, we observe

$$\|av^n\|_{H^2_q(\mathbb{X}_k)} \le \|a\|_{L_\infty(\mathbb{X}_k)}\|v\|_{H^2_q(\mathbb{X}_k)} + \|a\|_{W^1_\infty(\mathbb{X}_k)}\|v\|_{H^1_q(\mathbb{X}_k)} + \|a\|_{W^2_\infty(\mathbb{X}_k)}\|v\|_{L_q(\mathbb{X}_k)}.$$

By the estimates for $\|v\|_{H^s_q(\mathbb{X}_k)}$, $s \in \{0, 1\}$, from above, we obtain

$$\|av^n\|_{H^2_q(\mathbb{X}_k)} \le C\left(\|a\|_{L_\infty(\mathbb{X}_k)} + \|a\|_{W^1_\infty(\mathbb{X}_k)}|\lambda|^{-1/2} + \|a\|_{W^2_\infty(\mathbb{X}_k)}|\lambda|^{-1}\right)\|v\|_{H^2_{q,\lambda}(\mathbb{X}_k)}.$$

This shows that, for any given $\eta > 0$, choosing first $\|a\|_{L_\infty(\mathbb{X}_k)}$ sufficiently small and then $|\lambda|$ sufficiently large, we may achieve that

$$\|av^n\|_{H^2_q(\mathbb{X}_k)} + |\lambda|^{1/2}\|av^n\|_{H^1_q(\mathbb{X}_k)} + |\lambda|\|av^n\|_{L_q(\mathbb{X}_k)} \le \eta\|v\|_{H^2_{q,\lambda}(\mathbb{X}_k)}.$$

The estimates for the remaining boundary conditions can be derived in the same spirit and are therefore omitted. By a Neumann series argument as in case $k \in \mathcal{K}_0$, it follows that (3.7) holds true for the general geometry, with a possibly larger constant $C > 0$.

We split the solution $\bar{u}_k$ of (3.4), (3.5), (3.6) into $\bar{u}_k = \tilde{u}_k + \hat{u}_k$ in such a way that $\tilde{u}_k$ solves

$$L^\#_{\lambda,k}\tilde{u}_k = (\bar{f}_k, \bar{h}_{1,k}, \bar{h}_{2,k})^\mathsf{T}$$

and $\hat{u}_k$ solves

$$L^\#_{\lambda,k}\hat{u}_k = (P_k(u), \mathrm{tr}_{\mathbb{R}^{n-1}} Q_k(u), 0)^\mathsf{T}, \quad Q_k(u) = (Q^1_k(u), \dots, Q^{n-1}_k(u))$$

if $k \in \mathcal{K}_1$. For $k \in \mathcal{K}_0$, we introduce a similar decomposition $\bar{u}_k = \tilde{u}_k + \hat{u}_k$ with

$$L^\#_{\lambda,k}\tilde{u}_k = \bar{f}_k, \quad L^\#_{\lambda,k}\hat{u}_k = P_k(u).$$

For the solution u of (3.1) we therefore obtain

$$\begin{aligned} u = u_{(1)} + u_{(2)} &:= \mathcal{R}\left((\tilde{u})_{k\in\mathcal{K}}\right) + \mathcal{R}\left((\hat{u})_{k\in\mathcal{K}}\right) \\ &= \sum_{k\in\mathcal{K}} \xi_k\varphi^*_k\tilde{u}_k + \sum_{k\in\mathcal{K}} \xi_k\varphi^*_k\hat{u}_k. \end{aligned} \tag{3.8}$$

Employing (3.7) yields

$$\begin{aligned} &|\lambda|\|u_{(1)}\|_{L_q(\mathsf{M})} + \|u_{(1)}\|_{H^2_q(\mathsf{M})} \\ &\quad \le C\sum_{k\in\mathcal{K}}\|\bar{f}_k\|_{L_q(\mathbb{X}_k)} + C\sum_{k\in\mathcal{K}_1}\Big(\|\bar{H}_{1,k}\|_{H^1_q(\mathbb{X}_k)} + |\lambda|^{1/2}\|\bar{H}_{1,k}\|_{L_q(\mathbb{X}_k)} \\ &\qquad + \|\bar{H}_{2,k}\|_{H^2_q(\mathbb{X}_k)} + |\lambda|^{1/2}\|\bar{H}_{2,k}\|_{H^1_q(\mathbb{X}_k)} + |\lambda|\|\bar{H}_{2,k}\|_{L_q(\mathbb{X}_k)}\Big) \\ &\quad \le C\Big(\|f\|_{L_q(\mathsf{M})} + \|H_1\|_{H^1_q(\mathsf{M})} + |\lambda|^{1/2}\|H_1\|_{L_q(\mathsf{M})} \\ &\qquad + \|H_2\|_{H^2_q(\mathsf{M})} + |\lambda|^{1/2}\|H_2\|_{H^1_q(\mathsf{M})} + |\lambda|\|H_2\|_{L_q(\mathsf{M})}\Big) \end{aligned}$$

for all $\lambda \in \lambda_0 + \Sigma_{\pi-\phi}$, where $\bar{H}_{j,k} \in H_q^j(\mathbb{X}_k)$ are the localized versions of $H_j \in H_q^j(\mathsf{M})$. Here we have used the fact that if H_j is any extension of h_j from $W_q^{j-1/q}(\Sigma)$ to $H_q^j(\mathsf{M})$, then $\bar{H}_{j,k}$ is an extension of $\bar{h}_{j,k}$ from $W_q^{j-1/q}(\mathbb{R}^{n-1})$ to $H_q^j(\mathbb{X}_k)$.

To estimate $u_{(2)}$, note that (3.7) implies

$$
\begin{aligned}
&|\lambda|\|u_{(2)}\|_{L_q(\mathsf{M})} + \|u_{(2)}\|_{H_q^2(\mathsf{M})} \\
&\quad \le C \sum_{k\in\mathcal{K}} \|P_k(u)\|_{L_q(\mathbb{X}_k)} + C \sum_{k\in\mathcal{K}_1} \left(\|Q_k(u)\|_{H_q^1(\mathbb{X}_k)} + |\lambda|^{1/2} \|Q_k(u)\|_{L_q(\mathbb{X}_k)} \right) \\
&\quad \le C \left(\|u\|_{H_q^1(\mathsf{M})} + |\lambda|^{1/2} \|u\|_{L_q(\mathsf{M})} \right).
\end{aligned} \tag{3.9}
$$

By complex interpolation and Young's inequality, there exists a constant $C > 0$ such that

$$
\|u\|_{H_q^1(\mathsf{M})} \le C \|u\|_{L_q(\mathsf{M})}^{1/2} \cdot \|u\|_{H_q^2(\mathsf{M})}^{1/2} \le |\lambda|^{-1/2} C \left(|\lambda| \|u\|_{L_q(\mathsf{M})} + \|u\|_{H_q^2(\mathsf{M})} \right). \tag{3.10}
$$

Furthermore,

$$
|\lambda|^{1/2} \|u\|_{L_q(\mathsf{M})} \le |\lambda|^{-1/2} \left(|\lambda| \|u\|_{L_q(\mathsf{M})} + \|u\|_{H_q^2(\mathsf{M})} \right).
$$

By possibly further increasing $\lambda_0 > 0$, we can always achieve

$$
|\lambda|\|u_{(2)}\|_{L_q(\mathsf{M})} + \|u_{(2)}\|_{H_q^2(\mathsf{M})} \le \frac{1}{2} \left(|\lambda| \|u\|_{L_q(\mathsf{M})} + \|u\|_{H_q^2(\mathsf{M})} \right)
$$

for all $\lambda \in \lambda_0 + \Sigma_{\pi-\phi}$. Combining with the estimate for $u_{(1)}$, this yields (3.2) for all $\lambda \in \lambda_0 + \Sigma_{\pi-\phi}$. This estimate implies in particular that the operator L_λ defined in (3.3) has a left inverse S_λ, provided $\lambda \in \lambda_0 + \Sigma_{\pi-\phi}$.

We can even give an explicit formula for the left inverse S_λ. To this end, we use again (3.8), i.e.

$$
u = \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* \tilde{u}_k + \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* \hat{u}_k
$$

and define

$$
H_\lambda^\ell u := u_{(2)} = \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* \hat{u}_k.
$$

It follows from the considerations above that the linear operator

$$
H_\lambda^\ell : H_{q,\lambda}^2(\mathsf{M}; T\mathsf{M}) \to H_{q,\lambda}^2(\mathsf{M}; T\mathsf{M}), \quad u \mapsto u_{(2)},
$$

satisfies the norm estimate

$$
\|H_\lambda^\ell\| \le \frac{1}{2},
$$

provided $\lambda \in \lambda_0 + \Sigma_{\pi-\phi}$, where $H^2_{q,\lambda}(\mathsf{M}; T\mathsf{M})$ denotes the space $H^2_q(\mathsf{M}; T\mathsf{M})$ equipped with the norm $|\lambda|\,\|\cdot\|_{L_q(\mathsf{M})} + \|\cdot\|_{H^2_q(\mathsf{M})}$. By definition of $\tilde{u}_k$ we then obtain

$$u = \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* (L^{\#}_{\lambda,k})^{-1}(\bar{f}_k, \bar{h}_{1,k}, \bar{h}_{2,k}) + H^\ell_\lambda u,$$

where $(\bar{f}_k, \bar{h}_{1,k}, \bar{h}_{2,k}) = \bar{f}_k$ if $k \in \mathcal{K}_0$. Therefore, it follows that

$$S_\lambda(f, h_1, h_2) = u = (I - H^\ell_\lambda)^{-1} \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* (L^{\#}_{\lambda,k})^{-1}(\bar{f}_k, \bar{h}_{1,k}, \bar{h}_{2,k}).$$

It remains to prove the existence of a right inverse for the operator L_λ defined in (3.3). To this end, let

$$(f, h_1, h_2) \in L_q(\mathsf{M}; T\mathsf{M}) \times W^{1-1/q}_q(\Sigma; T\Sigma) \times W^{2-1/q}_q(\Sigma)$$

be given and define $u := S_\lambda(f, h_1, h_2) \in H^2_q(\mathsf{M}; T\mathsf{M})$ with the left inverse S_λ from above. In the sequel, we denote by

$$L_{\lambda,k} : H^2_q(\mathbb{X}_k; \mathbb{R}^n) \to L_q(\mathbb{X}_k; \mathbb{R}^n)$$

for $k \in \mathcal{K}_0$ and by

$$L_{\lambda,k} : H^2_q(\mathbb{X}_k; \mathbb{R}^n) \to L_q(\mathbb{X}_k; \mathbb{R}^n) \times W^{1-1/q}_q(\mathbb{R}^{n-1}; \mathbb{R}^{n-1}) \times W^{2-1/q}_q(\mathbb{R}^{n-1})$$

for $k \in \mathcal{K}_1$ the full operator L_λ from (3.3) in local coordinates, that is, $L_{\lambda,k}$ satisfies the relation $L_\lambda(\xi_k \varphi_k^* v) = \varphi_k^* L_{\lambda,k}(\psi_k^* \xi_k v)$ for $v \in H^2_q(\mathbb{X}_k; \mathbb{R}^n)$, where $\psi_k := \varphi_k^{-1}$. It follows that

$$L^1_k := L_{\lambda,k} - L^{\#}_{\lambda,k}$$

is of lower order, since the terms of highest order are already included in $L^{\#}_{\lambda,k}$. Applying L_λ to $u - H^\ell_\lambda u$ yields

$$\begin{aligned}
L_\lambda(u - H^\ell_\lambda u) &= L_\lambda \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* (L^{\#}_{\lambda,k})^{-1}(\bar{f}_k, \bar{h}_{1,k}, \bar{h}_{2,k}) \\
&= \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* L_{\lambda,k} (L^{\#}_{\lambda,k})^{-1}(\bar{f}_k, \bar{h}_{1,k}, \bar{h}_{2,k}) \\
&\quad + \sum_{k\in\mathcal{K}} \varphi_k^* [L_{\lambda,k}, \psi_k^* \xi_k] (L^{\#}_{\lambda,k})^{-1}(\bar{f}_k, \bar{h}_{1,k}, \bar{h}_{2,k}) \\
&= \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* (\bar{f}_k, \bar{h}_{1,k}, \bar{h}_{2,k}) \\
&\quad + \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* L^1_k (L^{\#}_{\lambda,k})^{-1}(\bar{f}_k, \bar{h}_{1,k}, \bar{h}_{2,k})
\end{aligned}$$

$$+\sum_{k\in\mathcal{K}}\varphi_k^*[L_{\lambda,k},\psi_k^*\xi_k](L^{\#}_{\lambda,k})^{-1}(\bar f_k,\bar h_{1,k},\bar h_{2,k}).$$

We note on the go that

$$\sum_{k\in\mathcal{K}}\xi_k\varphi_k^*(\bar f_k,\bar h_{1,k},\bar h_{2,k})=(f,h_1,h_2)$$

and we define

$$H^r_\lambda(f,h_1,h_2):=\sum_{k\in\mathcal{K}}\Big(\xi_k\varphi_k^*L_k^1(L^{\#}_{\lambda,k})^{-1}+\varphi_k^*[L_{\lambda,k},\psi_k^*\xi_k](L^{\#}_{\lambda,k})^{-1}\Big)(\bar f_k,\bar h_{1,k},\bar h_{2,k}).$$

Since both operators L_k^1 and $[L_{\lambda,k},\psi_k^*\xi_k]$ are of lower order, it follows that

$$\|H^r_\lambda\|\le\frac{1}{2}$$

for $\lambda\in\lambda_0+\Sigma_{\pi-\phi}$ and by possibly further increasing $\lambda_0>0$ if necessary, where the space

$$L_q(\mathsf{M};T\mathsf{M})\times W_q^{1-1/q}(\Sigma;T\Sigma)\times W_q^{2-1/q}(\Sigma)$$

is equipped with the norm on the right hand side of (3.2). This in turn implies that

$$(S_\lambda-H^\ell_\lambda S_\lambda)(I+H^r_\lambda)^{-1}$$

is a right inverse for L_λ. Hence, L_λ is invertible. □

In a next step, we consider homogeneous boundary conditions

$$\mathcal{P}_\Sigma\left((\nabla u-[\nabla u]^{\mathsf{T}})\nu_\Sigma\right)=0\quad\text{and}\quad(u|\nu_\Sigma)_g=0\quad\text{on }\Sigma\tag{3.11}$$

in (3.1) and we define an operator $L_{ps}:\mathsf{D}(L_{ps})\to L_q(\mathsf{M};T\mathsf{M})$ by

$$L_{ps}u=-\Delta_{\mathsf{M}}u+\mathsf{Ric}^\sharp u,\quad u\in\mathsf{D}(L_{ps}):=\{u\in H_q^2(\mathsf{M};T\mathsf{M}):u\text{ satisfies (3.11)}\}.$$

Note that by Proposition 3.1, the operator $(\lambda+L_{ps})$ is invertible for any $\lambda\in\lambda_0+\Sigma_{\pi-\phi}$.

We can then show the following stronger result.

Proposition 3.2. *There exists $\omega_0>0$ such that for all $\omega>\omega_0$*

$$\omega+L_{ps}\in H^\infty(L_q(\mathsf{M};T\mathsf{M}))\textit{ with }H^\infty\textit{-angle }<\pi/2.\tag{3.12}$$

Proof. We define the linear operator $L_k:\mathsf{D}(L_k)\to L_q(\mathbb{X}_k;\mathbb{R}^n)$ by

$$L_ku:=-\bar g^{ij}_{(k)}\partial_i\partial_j u\quad\text{in }\mathbb{X}_k$$

and for $k\in\mathcal{K}_1$,

$$T_k u := \begin{pmatrix} \mathrm{tr}_{\mathbb{R}^{n-1}}\big(\frac{1}{\sqrt{\bar{g}^{nn}_{(k)}}}\bar{g}^{nj}_{(k)}\partial_j u^1 - \frac{\bar{g}^{nj}_{(k)}\bar{g}^{n1}_{(k)}}{(\bar{g}^{nn}_{(k)})^{3/2}}\partial_j u^n\big) \\ \vdots \\ \mathrm{tr}_{\mathbb{R}^{n-1}}\big(\frac{1}{\sqrt{\bar{g}^{nn}_{(k)}}}\bar{g}^{nj}_{(k)}\partial_j u^{n-1} - \frac{\bar{g}^{nj}_{(k)}\bar{g}^{n,n-1}_{(k)}}{(\bar{g}^{nn}_{(k)})^{3/2}}\partial_j u^n\big) \\ \mathrm{tr}_{\mathbb{R}^{n-1}}\frac{1}{\sqrt{\bar{g}^{nn}_{(k)}}}u^n \end{pmatrix},$$

where

$$\mathsf{D}(L_k) = \begin{cases} H^2_q(\mathbb{R}^n;\mathbb{R}^n) & \text{if } k \in \mathcal{K}_0, \\ \{u \in H^2_q(\mathbb{R}^n_+;\mathbb{R}^n) : T_k u = 0 \text{ on } \mathbb{R}^{n-1}\} & \text{if } k \in \mathcal{K}_1. \end{cases}$$

We claim that there exists some $\omega_0 > 0$ and $\phi^\infty \in (0, \pi/2)$ such that

$$\omega + L_k \in H^\infty(L_q(\mathbb{X}_k;\mathbb{R}^n)) \text{ with } H^\infty\text{-angle } < \phi^\infty \quad \text{for all } \omega > \omega_0. \tag{3.13}$$

It is well-known that (3.13) holds true for $k \in \mathcal{K}_0$, see for instance [9, Theorem 6.1] or [7, Theorem 4.1], as long as $R > 0$ is sufficiently small.

In the case of $k \in \mathcal{K}_1$, in the planar case, i.e. $G_k = I_n$, one can check that

$$L_k = -\mathrm{diag}\,[\Delta_N, \cdots, \Delta_N, \Delta_D] : \mathsf{D}(L_k) \to L_p(\mathbb{X}_k;\mathbb{R}^n),$$

where Δ_N and Δ_D are the Neumann and Dirichlet Laplacian in $\mathbb{R}^n_+$, respectively. Then it follows from well-known results that (3.13) holds, see [8, Theorem 7.4]. For a general geometry, using a similar perturbation argument to that in [11], one can show that, by making $R > 0$ sufficiently small, (3.13) is at our disposal.

We seek to find an expression for the resolvent $(\lambda + L_{ps})^{-1}$. To this end, consider the splitting (3.8) for the solution u of (3.1) with homogeneous boundary conditions. This yields

$$\begin{aligned} (\lambda + L_{ps})^{-1} f = u = u_{(1)} + u_{(2)} &:= \mathcal{R}\left((\tilde{u})_{k\in\mathcal{K}}\right) + \mathcal{R}\left((\hat{u})_{k\in\mathcal{K}}\right) \\ &= \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* \tilde{u}_k + \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* \hat{u}_k \\ &= \mathcal{R}\left(((\lambda + L_k)^{-1}\bar{f}_k)_{k\in\mathcal{K}}\right) + R(\lambda)(f), \end{aligned} \tag{3.14}$$

where

$$R(\lambda)(f) := \sum_{k\in\mathcal{K}_0} \xi_k \varphi_k^* (\lambda + L_k)^{-1} P_k(u) + \sum_{k\in\mathcal{K}_1} \xi_k \varphi_k^* (L^\#_{\lambda,k})^{-1}(P_k(u), \mathrm{tr}_{\mathbb{R}^{n-1}} Q_k(u), 0)^\mathsf{T}.$$

The estimates (3.9), (3.10) and (3.2) then yield the existence of $\omega_0 > 0$ such that

$$\|R(\lambda) f\|_{L_q(\mathsf{M})} \le C|\lambda|^{-3/2}\|f\|_{L_q(\mathsf{M})} \tag{3.15}$$

for all $\lambda \in \omega + \Sigma_{\pi-\phi^\infty}$, $\omega \ge \omega_0$.

Note that the H^∞-bound K_{ϕ^∞}, cf. (D.1), can be chosen uniformly for $\omega + L_k$, where $\omega > \omega_0$ by possibly further increasing ω_0. Given any $h \in \mathcal{H}_0(\Sigma_{\pi-\phi^\infty})$, see Appendix D for a definition of $\mathcal{H}_0(\Sigma_\phi)$, in view of (3.14) we obtain

$$\begin{aligned}
&\|h(\omega + L_{ps})f\|_{L_q(\mathsf{M})} \\
&= \left\| \frac{1}{2\pi i} \int_\Gamma h(\lambda)(\lambda + \omega + L_{ps})^{-1} f \, d\lambda \right\|_{L_q(\mathsf{M})} \\
&\le c \left\| \frac{1}{2\pi i} \sum_{k\in\mathcal{K}} \xi_k \varphi_k^* \left[\int_\Gamma h(\lambda)(\lambda + \omega + L_k)^{-1} \bar{f}_k \, d\lambda \right] \right\|_{L_q(\mathsf{M})} + c \left\| \int_\Gamma h(\lambda) R(\lambda) f \, d\lambda \right\|_{L_q(\mathsf{M})} \\
&\le M \sum_{k\in\mathcal{K}} \left\| \frac{1}{2\pi i} \int_\Gamma h(\lambda)(\lambda + \omega + L_k)^{-1} \bar{f}_k \, d\lambda \right\|_{L_q(\mathbb{X}_k)} + M\|h\|_\infty \int_\Gamma \|R(\lambda) f\|_{L_q(\mathsf{M})} \, ds \\
&\le M\|h\|_\infty \sum_{k\in\mathcal{K}} \|\bar{f}_k\|_{L_q(\mathbb{X}_k)} + M\|h\|_\infty \|f\|_{L_q(\mathsf{M})} \qquad (3.16) \\
&\le M\|h\|_\infty \|f\|_{L_q(\mathsf{M})},
\end{aligned}$$

where the integral contour Γ is defined as in (D.2) and (3.16) follows from (3.13) and (3.15). □

We have shown in Proposition 3.1 that, for every $\lambda \in \omega + \Sigma_{\pi-\phi^\infty}$ and $f \in L_q(\mathsf{M}; T\mathsf{M})$, the equation $\lambda u - L_{ps}u = f$ has a unique solution $u \in \mathsf{D}(L_{ps})$. More can be said about the solution u if, in addition, $f \in L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$.

Proposition 3.3. *There exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ and $f \in L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$, the equation $\lambda u - L_{ps}u = f$ has a unique solution $u \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})$.*

Proof. We consider the Neumann problem

$$\begin{cases} \Delta_B \phi = \lambda \operatorname{div} u & \text{on} \quad \mathsf{M}, \\ (\operatorname{grad} \phi | \nu_\Sigma)_g = 0 & \text{on} \quad \Sigma, \end{cases} \tag{3.17}$$

where Δ_B is the Laplace-Beltrami operator on (M, g). By Lemma B.6, (3.17) has a unique (up to a constant) solution $\phi \in H^3_q(\mathsf{M})$. Employing Lemma B.1 repeatedly, we obtain

$$\begin{aligned}
&\|\operatorname{grad} \phi\|^2_{L_2(\mathsf{M})} \\
&= (-\Delta_B \phi | \phi)_\mathsf{M} = -\lambda(\operatorname{div} u | \phi)_\mathsf{M} = \lambda(u | \operatorname{grad} \phi)_\mathsf{M} \\
&= (\Delta_\mathsf{M} u | \operatorname{grad} \phi)_\mathsf{M} + (f | \operatorname{grad} \phi)_\mathsf{M} - (\mathrm{Ric}^\sharp u | \operatorname{grad} \phi)_\mathsf{M} \\
&= -(\nabla u | \nabla \operatorname{grad} \phi)_\mathsf{M} + (\nabla_{\nu_\Sigma} u | \operatorname{grad} \phi)_\Sigma - (\mathrm{Ric}^\sharp u | \operatorname{grad} \phi)_\mathsf{M} \qquad (3.18) \\
&= (u | \Delta_\mathsf{M} \operatorname{grad} \phi)_\mathsf{M} + (\nabla_{\nu_\Sigma} u | \operatorname{grad} \phi)_\Sigma - (u | \nabla_{\nu_\Sigma} \operatorname{grad} \phi)_\Sigma - (\mathrm{Ric}^\sharp u | \operatorname{grad} \phi)_\mathsf{M}
\end{aligned}$$

$$= (u|\mathsf{grad}\,\Delta_B\phi)_{\mathsf{M}} + (\nabla_{\nu_\Sigma} u|\mathsf{grad}\,\phi)_\Sigma - (u|\nabla_{\nu_\Sigma}\mathsf{grad}\,\phi)_\Sigma \tag{3.19}$$

$$= -\lambda\|\mathsf{div}\,u\|^2_{L_2(\mathsf{M})} + (\nabla_{\nu_\Sigma} u|\mathsf{grad}\,\phi)_\Sigma - (u|\nabla_{\nu_\Sigma}\mathsf{grad}\,\phi)_\Sigma. \tag{3.20}$$

By the definition of $\mathbb{P}_H$ and the fact that $f \in L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$, we conclude that

$$(f|\mathsf{grad}\,\phi)_{\mathsf{M}} = (\mathbb{P}_H f|\mathsf{grad}\,\phi)_{\mathsf{M}} = (f - \mathsf{grad}\,\psi_f|\mathsf{grad}\,\phi)_{\mathsf{M}} = 0.$$

Therefore, (3.18) follows. In (3.19), we have used the property

$$\Delta_{\mathsf{M}}\mathsf{grad}\,\phi = \mathsf{grad}\,\Delta_B\phi + \mathsf{Ric}^\sharp\mathsf{grad}\,\phi, \tag{3.21}$$

see Lemma B.2.

Since ϕ is a scalar function, we have $\nabla\mathsf{grad}\,\phi = (\nabla\mathsf{grad}\,\phi)^\mathsf{T}$. Employing (2.8), with u replaced by $\mathsf{grad}\,\phi$, we then obtain

$$\mathcal{P}_\Sigma\nabla_{\nu_\Sigma}\mathsf{grad}\,\phi = \mathcal{P}_\Sigma(\nabla\mathsf{grad}\,\phi)\nu_\Sigma = \mathcal{P}_\Sigma(\nabla\mathsf{grad}\,\phi)^\mathsf{T}\nu_\Sigma = L_\Sigma\mathsf{grad}\,\phi.$$

Hence, the last term on the RHS of (3.20) can be rewritten as

$$(u|\nabla_{\nu_\Sigma}\mathsf{grad}\,\phi)_\Sigma = (u|\mathcal{P}_\Sigma\nabla_{\nu_\Sigma}\mathsf{grad}\,\phi)_\Sigma = (L_\Sigma u|\mathsf{grad}\,\phi)_\Sigma$$

in view of (3.11). We thus infer that

$$\begin{aligned}\|\mathsf{grad}\,\phi\|^2_{L_2(\mathsf{M})} + \lambda\|\mathsf{div}\,u\|^2_{L_2(\mathsf{M})} &= (\nabla_{\nu_\Sigma}u - L_\Sigma u|\mathsf{grad}\,\phi)_\Sigma \\ &= ((\nabla u - [\nabla u]^\mathsf{T})\nu_\Sigma|\mathsf{grad}\,\phi)_\Sigma = 0,\end{aligned}$$

where we have used (3.11) once more. We thus have $\mathsf{div}\,u = 0$ and $(u|\nu_\Sigma)_g = 0$ and this, in turn, implies $u \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})$, in virtue of Lemma B.6. □

Proposition 3.3 reveals that for $\lambda > \lambda_0$

$$(\lambda + L_{ps})^{-1}\mathsf{R}(\mathbb{P}_H) \subseteq \mathsf{R}(\mathbb{P}_H). \tag{3.22}$$

We will further show that

$$(\lambda + L_{ps})^{-1}\mathsf{N}(\mathbb{P}_H) \subseteq \mathsf{N}(\mathbb{P}_H). \tag{3.23}$$

Indeed, if $f = \mathsf{grad}\,g$ for some $g \in H^1_q(\mathsf{M})$, then we consider

$$\begin{cases} (\lambda - \Delta_B)\phi = g & \text{on} \quad \mathsf{M}, \\ (\mathsf{grad}\,\phi|\nu_\Sigma)_g = 0 & \text{on} \quad \Sigma. \end{cases} \tag{3.24}$$

For sufficiently large $\lambda_0 > 0$ and all $\lambda > \lambda_0$, (3.24) has a unique solution $\phi \in H^3_q(\mathsf{M})$ by means of a localization argument as in Section 3.1. Let $v = \mathsf{grad}\,\phi$. Then v satisfies

$$\lambda v - \operatorname{grad} \Delta_B \phi = \operatorname{grad} g = f.$$

Using (3.21) once more, it is easy task to check that v solves

$$\begin{cases} (\lambda - \Delta_{\mathsf{M}} + \mathsf{Ric}^\sharp)v = f & \text{on} \quad \mathsf{M}, \\ (v|\nu_\Sigma)_g = 0 & \text{on} \quad \Sigma. \end{cases}$$

On the other hand, since $\nabla v = [\nabla v]^{\mathsf{T}}$, the boundary condition

$$\mathcal{P}_\Sigma \left((\nabla v - [\nabla v]^{\mathsf{T}})\nu_\Sigma \right) = 0 \quad \text{on } \Sigma$$

is automatically satisfied. Hence v is indeed a solution of (3.1). Uniqueness of solutions of (3.1) implies that $v = u$ and thus (3.23) is proved.

By (3.22) and (3.23), given any $u \in \mathsf{D}(L_{ps})$, one has

$$L_{ps}\mathbb{P}_H u \in \mathsf{R}(\mathbb{P}_H) \quad \text{and} \quad L_{ps}(I - \mathbb{P}_H)u \in \mathsf{N}(\mathbb{P}_H). \tag{3.25}$$

Now we are ready to study the *strong surface Stokes operator with perfect slip boundary conditions* $A_{ps} : \mathsf{D}(A_{ps}) \to L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$, defined by

$$A_{ps}u = -\mu_s \mathbb{P}_H(\Delta_{\mathsf{M}} u + \mathsf{Ric}^\sharp u)$$

with

$$\mathsf{D}(A_{ps}) = \{u \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}) : (u|\nu_\Sigma)_g = 0, \;\; \mathcal{P}_\Sigma \left((\nabla u - [\nabla u]^{\mathsf{T}})\nu_\Sigma \right) = 0 \text{ on } \Sigma\}.$$

In spite of (2.2), we include the condition $(u|\nu_\Sigma)_g = 0$ for extra emphasis.

Let $\widetilde{A}_{ps} := A_{ps} + 2\mu_s \mathbb{P}_H \mathsf{Ric}^\sharp : \mathsf{D}(A_{ps}) \to L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$. Since for any $u \in \mathsf{D}(L_{ps})$, one can deduce

$$\mathbb{P}_H L_{ps} u = \mathbb{P}_H L_{ps} \mathbb{P}_H u + \mathbb{P}_H L_{ps}(I - \mathbb{P}_H)u = L_{ps}\mathbb{P}_H u$$

from (3.25), it holds that

$$\widetilde{A}_{ps} = \mu_s L_{ps}|_{\mathsf{D}(A_{ps})}.$$

Therefore, for sufficiently large $\omega > 0$, $\omega + \widetilde{A}_{ps} \in H^\infty(L_{q,\sigma}(\mathsf{M}; T\mathsf{M}))$, by (3.12). By possibly enlarging $\omega_0 > 0$, the following theorem is an immediate consequence of [28, Corollary 3.3.15].

Theorem 3.4. *There exists $\omega_0 > 0$ such that for all $\omega > \omega_0$*

$$\omega + A_{ps} \in H^\infty(L_{q,\sigma}(\mathsf{M}; T\mathsf{M})) \text{ with } H^\infty\text{-angle } < \pi/2. \tag{3.26}$$

3.2. Weak formulation

For notational brevity, let $A_0 = \omega + A_{ps}$, $\omega > \omega_0$ with ω_0 being defined in (3.26). Note that $\omega + A_{ps}$ is invertible. We set

$$Z_0 = X_0 = L_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \quad \text{and} \quad Z_1 := \mathsf{D}(A_{ps}).$$

By [2, Theorems V.1.5.1 and V.1.5.4], the pair (Z_0, A_0) generates an interpolation-extrapolation scale (Z_β, A_β), $\beta \in \mathbb{R}$, with respect to the complex interpolation functor. In particular, when $\beta \in (0,1)$, A_β is the Z_β-realization of A_0, where

$$Z_\beta = \mathsf{D}(A_0^\beta) = [Z_0, Z_1]_\beta$$

due to (3.26). Let $Z_0^\sharp := (Z_0)' = L_{q',\sigma}(\mathsf{M}; T\mathsf{M})$ and

$$\begin{aligned} A_0^\sharp &:= (A_0)' = \omega - \mu_s \mathbb{P}_H(\Delta_{\mathsf{M}} - \mathsf{Ric}^\sharp) : \mathsf{D}(A_0^\sharp) \to Z_0^\sharp, \\ \mathsf{D}(A_0^\sharp) = Z_1^\sharp &:= \{u \in H^2_{q',\sigma}(\mathsf{M}; T\mathsf{M}) : \mathcal{P}_\Sigma\left((\nabla u - [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) = 0 \text{ on } \Sigma\}. \end{aligned}$$

Then $(Z_0^\sharp, A_0^\sharp)$ generates an interpolation-extrapolation scale $(Z_\beta^\sharp, A_\beta^\sharp)$, $\beta \in \mathbb{R}$, the dual scale. By [2, Theorem V.1.5.12], it holds that

$$(Z_\beta)' = Z^\sharp_{-\beta} \quad \text{and} \quad (A_\beta)' = A^\sharp_{-\beta} \tag{3.27}$$

for $\beta \in \mathbb{R}$. Particularly, when $\beta = -1/2$, the operator $A_{-1/2} : Z_{1/2} \to Z_{-1/2}$ satisfies

$$\mathsf{D}(A_{-1/2}) = Z_{1/2} = [Z_0, Z_1]_{1/2} = H^1_{q,\sigma}(\mathsf{M}; T\mathsf{M}),$$

see Proposition C.6, and $Z_{-1/2} = (Z^\sharp_{1/2})'$. Note that

$$Z^\sharp_{1/2} = [Z_0^\sharp, Z_1^\sharp]_{1/2} = H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M}).$$

Therefore,

$$Z_{1/2} = X_{1/2} \quad \text{and} \quad Z_{-1/2} = X_{-1/2},$$

where $X_{1/2}$ and $X_{-1/2}$ were introduced in (2.11). By the definitions in (2.3), one can follow the arguments in [32, Propositions 2.3 and 2.4] and show that for any $\theta \in (0,1)$

$$[Z_{-1/2}, Z_{1/2}]_\theta = H^{2\theta-1}_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \quad (Z_{-1/2}, Z_{1/2})_{\theta,p} = B^{2\theta-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M}), \tag{3.28}$$

see also Proposition C.5. By replacing q by q', we infer from Section 3.1 that $A_0^\sharp \in H^\infty(Z_0^\sharp)$ with H^∞-angle $\phi^\infty_{A_0^\sharp} < \pi/2$. Since $A^\sharp_{1/2}$ is the $Z^\sharp_{1/2}$-realization of $A_0^\sharp$, it follows from [28, Proposition 3.3.14] and (3.27) that

$$A_{-1/2} = (A^\sharp_{1/2})' \in H^\infty(Z_{-1/2}) \text{ with } H^\infty\text{-angle} < \pi/2.$$

We call the operator $A_{-1/2} : Z_{1/2} \to Z_{-1/2}$ the *weak surface Stokes operator with perfect slip boundary conditions*.

Since $A_{-1/2}$ is the closure of A_0 in $Z_{-1/2}$, it follows that $A_{-1/2}u = A_0 u$ for all $u \in \mathsf{D}(A_0) = Z_1$. Thus for any $v \in Z^\sharp_{1/2}$, it follows from (2.8), (3.11) and Lemma B.1 (b) (ii) that

$$\begin{aligned}
\langle A_{-1/2}u|v\rangle_{\mathsf{M}} &= (A_0 u|v)_{\mathsf{M}} \\
&= \omega(u|v)_{\mathsf{M}} + \mu_s(\nabla u|\nabla v)_{\mathsf{M}} - \mu_s(\mathsf{Ric}^\sharp u|v)_{\mathsf{M}} - \mu_s(\nabla u \nu_\Sigma|v)_\Sigma \\
&= \omega(u|v)_{\mathsf{M}} + \mu_s(\nabla u|\nabla v)_{\mathsf{M}} - \mu_s(\mathsf{Ric}^\sharp u|v)_{\mathsf{M}} - \mu_s(\mathcal{P}_\Sigma([\nabla u]^{\mathsf{T}}\nu_\Sigma)|v)_\Sigma \\
&= \omega(u|v)_{\mathsf{M}} + \mu_s(\nabla u|\nabla v)_{\mathsf{M}} - \mu_s(\mathsf{Ric}^\sharp u|v)_{\mathsf{M}} - \mu_s(L_\Sigma u|v)_\Sigma.
\end{aligned}$$

By the density of Z_1 in $Z_{1/2}$, we infer that for all

$$\begin{aligned}
(u, v) \in Z_{1/2} \times Z^\sharp_{1/2} &= H^1_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \times H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M}), \\
\langle A_{-1/2}u|v\rangle_{\mathsf{M}} &= \omega(u|v)_{\mathsf{M}} + \mu_s(\nabla u|\nabla v)_{\mathsf{M}} - \mu_s(\mathsf{Ric}^\sharp u|v)_{\mathsf{M}} - \mu_s(L_\Sigma u|v)_\Sigma. \qquad (3.29)
\end{aligned}$$

4. H^∞-calculus of surface Stokes operator with Navier boundary conditions

Recall the definition of the *weak Stokes operator with Navier boundary conditions* $A^{\mathsf{w}}_N : X_{1/2} \to X_{-1/2}$ provided in Section 2. In view of (3.29), easy computations show that

$$\langle (\omega + A^{\mathsf{w}}_N)u|v\rangle_{\mathsf{M}} = \langle A_{-1/2}u|v\rangle_{\mathsf{M}} + 2\mu_s(L_\Sigma u|v)_\Sigma + \alpha\mu_s(u|v)_\Sigma$$

is valid for all $(u, v) \in H^1_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \times H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})$. We define the operator $B_N : \mathsf{D}(B_N) \to X_{-1/2}$ by

$$\langle B_N u|v\rangle_{\mathsf{M}} = 2\mu_s(L_\Sigma u|v)_\Sigma + \alpha\mu_s(u|v)_\Sigma$$

for all $(u, v) \in \mathsf{D}(B_N) \times H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})$. The domain $\mathsf{D}(B_N)$ will be specified in the following calculations. By trace theory and Hölder's inequality, we have

$$|\langle B_N u|v\rangle_{\mathsf{M}}| \le C\|u\|_{L_q(\Sigma)}\|v\|_{L_{q'}(\Sigma)} \le C\|u\|_{H^s_{q,\sigma}(\mathsf{M})}\|v\|_{H^1_{q',\sigma}(\mathsf{M})}$$

for any $s > 1/q$. Thus, by choosing $\mathsf{D}(B_N) = H^s_{q,\sigma}(\mathsf{M}; T\mathsf{M}) = [X_{-1/2}, X_{1/2}]_\theta$ with $s \in (1/q, 1)$ and $\theta = (s+1)/2$, $B_N \in \mathcal{L}(\mathsf{D}(B_N), X_{-1/2})$ and thus is a lower order perturbation of $A_{-1/2} : X_{1/2} \to X_{-1/2}$. Then it again follows from [28, Corollary 3.3.15] that, by possibly enlarging $\omega_0 > 0$,

$$\omega + A^{\mathsf{w}}_N \in H^\infty(X_{-1/2}) \text{ with } H^\infty\text{-angle } < \pi/2 \quad \text{for all } \omega > \omega_0. \qquad (4.1)$$

Next, we will show that A_N also admits bounded H^∞-calculus. Take $\omega > 0$ sufficiently large so that $\omega + A^{\mathsf{w}}_N$ is invertible. Given any $u \in \mathsf{D}(A_N)$, see (2.4) and (2.5), there exists a unique

$w \in X_{1/2}$ such that $(\omega + A_N)u = (\omega + A_N^{\mathsf{w}})w$. Because of (2.10), for all $v \in H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})$ we have

$$((\omega + A_N)u|v)_{\mathsf{M}} = \langle(\omega + A_N^{\mathsf{w}})w|v\rangle_{\mathsf{M}} = \langle(\omega + A_N^{\mathsf{w}})u|v\rangle_{\mathsf{M}}.$$

The injectivity of $\omega + A_N^{\mathsf{w}}$ implies that $u = w$. Thus, $\mathrm{gr}(A_N) \subset \mathrm{gr}(A_N^{\mathsf{w}})$, where $\mathrm{gr}(\mathcal{A})$ is the graph of an operator $\mathcal{A}$ in $X_{-1/2}$. Lemma 4.1 further implies that $\omega + A_N : \mathsf{D}(A_N) \subset L_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \to L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$ is closed and bijective. Therefore, $\omega + A_N$ is the $L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$-realization of $\omega + A_N^{\mathsf{w}}$ and it inherits the bounded H^∞-calculus property, i.e., there exists $\omega_0 > 0$ such that

$$\omega + A_N \in H^\infty(X_0) \text{ with } H^\infty\text{-angle } < \pi/2 \quad \text{for all } \omega > \omega_0. \tag{4.2}$$

Lemma 4.1. *There exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ and $f \in L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$*

$$\begin{cases} (\lambda - \mathbb{P}_H \Delta_{\mathsf{M}} - \mathbb{P}_H \mathsf{Ric}^\sharp)u = f & \text{on} \quad \mathsf{M}, \\ \alpha u + \mathcal{P}_\Sigma\left((\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) = 0 & \text{on} \quad \Sigma, \\ (u|\nu_\Sigma)_g = 0 & \text{on} \quad \Sigma \end{cases} \tag{4.3}$$

has a unique solution $u \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})$.

Proof. We proved in Proposition 3.1 that there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, for all $f \in L_q(\mathsf{M}; T\mathsf{M})$ and all $h \in W_q^{1-1/q}(\Sigma; T\Sigma)$ there exists a unique solution $u \in H^2_q(\mathsf{M}; T\mathsf{M})$ of the problem

$$\begin{cases} (\lambda - \Delta_{\mathsf{M}} + \mathsf{Ric}^\sharp)u = f & \text{on} \quad \mathsf{M}, \\ \mathcal{P}_\Sigma\left((\nabla u - [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) = h & \text{on} \quad \Sigma, \\ (u|\nu_\Sigma)_g = 0 & \text{on} \quad \Sigma, \end{cases} \tag{4.4}$$

and, in addition, there exists a constant $C = C(\lambda_0) > 0$ such that the estimate

$$\lambda\|u\|_{L_q(\mathsf{M})} + \|u\|_{H^2_q(\mathsf{M})} \leq C\left(\|f\|_{L_q(\mathsf{M})} + \lambda^{1/2}\|H\|_{L_q(\mathsf{M})} + \|H\|_{H^1_q(\mathsf{M})}\right) \tag{4.5}$$

holds for the solution $u \in H^2_q(\mathsf{M}; T\mathsf{M})$ of (4.4), where H is any extension of h from $W_q^{1-1/q}(\Sigma)$ to $H^1_q(\mathsf{M})$.

In a first step, we will show that there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, for all $f \in L_q(\mathsf{M}; T\mathsf{M})$ and all $h \in W_q^{1-1/q}(\Sigma; T\Sigma)$ there exists a unique solution $(v, \pi) \in H^2_q(\mathsf{M}; T\mathsf{M}) \times \dot{H}^1_q(\mathsf{M})$ of the Stokes problem

$$\begin{cases} (\lambda - \Delta_{\mathsf{M}} + \mathsf{Ric}^\sharp)v + \mathsf{grad}\,\pi = f & \text{on} \quad \mathsf{M}, \\ \mathsf{div}\, v = 0 & \text{on} \quad \mathsf{M}, \\ \mathcal{P}_\Sigma\left((\nabla v - [\nabla v]^{\mathsf{T}})\nu_\Sigma\right) = h & \text{on} \quad \Sigma, \\ (v|\nu_\Sigma)_g = 0 & \text{on} \quad \Sigma, \end{cases} \tag{4.6}$$

satisfying the estimate

$$\lambda\|v\|_{L_q(\mathsf{M})}+\|v\|_{H_q^2(\mathsf{M})}+\|\mathsf{grad}\,\pi\|_{L_q(\mathsf{M})}\le C\left(\|f\|_{L_q(\mathsf{M})}+\lambda^{1/2}\|H\|_{L_q(\mathsf{M})}+\|H\|_{H_q^1(\mathsf{M})}\right). \tag{4.7}$$

Indeed, let $v:=\mathbb{P}_H u=u-\mathsf{grad}\,\psi_u$, where $u\in H_q^2(\mathsf{M};T\mathsf{M})$ is the unique solution of (4.4) and $\mathsf{grad}\,\psi_u\in H_q^2(\mathsf{M};T\mathsf{M})$ is the unique solution of

$$\begin{cases}\Delta_B\psi_u=\mathsf{div}\,u & \text{on} \quad \mathsf{M},\\ (\mathsf{grad}\,\psi_u|\nu_\Sigma)_g=0 & \text{on} \quad \Sigma,\end{cases}$$

see Lemma B.6. Defining $\pi:=\lambda\psi_u-\mathsf{div}\,u$, it follows from (3.21) that the pair (v,π) is a solution of (4.6). Moreover, by (B.14) and (4.5), we have the estimates

$$\begin{aligned}\|\mathsf{grad}\,\psi_u\|_{H_q^2(\mathsf{M})}&\le C\|u\|_{H_q^2(\mathsf{M})}\\ &\le C\left(\|f\|_{L_q(\mathsf{M})}+\lambda^{1/2}\|H\|_{L_q(\mathsf{M})}+\|H\|_{H_q^1(\mathsf{M})}\right)\end{aligned}$$

and

$$\begin{aligned}\|\mathsf{grad}\,\pi\|_{L_q(\mathsf{M})}&\le\lambda\|\mathsf{grad}\,\psi_u\|_{L_q(\mathsf{M})}+\|\mathsf{grad}\,\mathsf{div}\,u\|_{L_q(\mathsf{M})}\\ &\le C\left(\lambda\|u\|_{L_q(\mathsf{M})}+\|u\|_{H_q^2(\mathsf{M})}\right)\\ &\le C\left(\|f\|_{L_q(\mathsf{M})}+\lambda^{1/2}\|H\|_{L_q(\mathsf{M})}+\|H\|_{H_q^1(\mathsf{M})}\right),\end{aligned}$$

and therefore, the functions (v,π) satisfy the estimate (4.7).

Uniqueness of the solution (v,π) to (4.6) can be seen as follows. Assume that $f=0$ and $h=0$ in (4.6). Then $u:=\mathbb{P}_H v=v$ solves (4.4) with $f=0$ and $h=0$, as $\mathbb{P}_H L_{ps}=L_{ps}\mathbb{P}_H$ (see Section 3) and $\mathbb{P}_H\mathsf{grad}\,\pi=0$. Since the solution to (4.4) is unique, it follows that $v=u=0$. Inserting $v=0$ into (4.6) we obtain $\mathsf{grad}\,\pi=0$, and therefore $\pi=0$ in $\dot H_q^1(\mathsf{M})$.

Having the unique solvability of (4.6) and the estimate (4.7) at hand, we may apply a perturbation argument as in the proof of Proposition 3.1 in order to replace the left hand side of $(4.6)_1$ by

$$\lambda v-\Delta_{\mathsf{M}}v-\mathsf{Ric}^\sharp v=(\lambda v-\Delta_{\mathsf{M}}v+\mathsf{Ric}^\sharp v)-2\mathsf{Ric}^\sharp v$$

and the boundary condition $(4.6)_3$ by

$$\mathcal{P}_\Sigma\left((\nabla v+[\nabla v]^{\mathsf{T}})\nu_\Sigma\right)+\alpha v=\mathcal{P}_\Sigma\left((\nabla v-[\nabla v]^{\mathsf{T}})\nu_\Sigma\right)+2L_\Sigma v+\alpha v.$$

Indeed, for $v\in H_q^2(\mathsf{M};T\mathsf{M})$ it holds that

$$\|\mathsf{Ric}^\sharp v\|_{L_q(\mathsf{M})}\le C\|v\|_{L_q(\mathsf{M})}\le\lambda^{-1}C\left(\lambda\|v\|_{L_q(\mathsf{M})}+\|v\|_{H_q^2(\mathsf{M})}\right).$$

Concerning the boundary condition, we observe that $2L_\Sigma v + \alpha v \in W_q^{2-1/q}(\Sigma)$ for $v \in H_q^2(\mathsf{M}; T\mathsf{M})$. Hence, there exists an extension $Q(v) \in H_q^2(\mathsf{M})$ of $2L_\Sigma v + \alpha v$ such that

$$\|Q(v)\|_{H_q^s(\mathsf{M})} \leq C(s)\|v\|_{H_q^s(\mathsf{M})}, \quad v \in H_q^2(\mathsf{M}; T\mathsf{M}),\ s \in [0,2].$$

Then, by complex interpolation and Young's inequality,

$$\|Q(v)\|_{H_q^1(\mathsf{M})} \leq C\|v\|_{H_q^1(\mathsf{M})} \leq \lambda^{-1/2} C\left(\lambda\|v\|_{L_q(\mathsf{M})} + \|v\|_{H_q^2(\mathsf{M})}\right),$$

and

$$\lambda^{1/2}\|Q(v)\|_{L_q(\mathsf{M})} \leq \lambda^{1/2} C\|v\|_{L_q(\mathsf{M})} \leq \lambda^{-1/2} C\left(\lambda\|v\|_{L_q(\mathsf{M})} + \|v\|_{H_q^2(\mathsf{M})}\right).$$

Therefore, a very similar Neumann series argument as in the proof of Proposition 3.1 yields the existence of a number $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ and for all $f \in L_q(\mathsf{M}; T\mathsf{M})$ there exists a unique solution $(u, \pi) \in H_q^2(\mathsf{M}; T\mathsf{M}) \times \dot{H}_q^1(\mathsf{M})$ of the problem

$$\begin{cases} (\lambda - \Delta_{\mathsf{M}} - \mathsf{Ric}^\sharp)u + \mathsf{grad}\,\pi = f & \text{on} \quad \mathsf{M}, \\ \mathsf{div}\,u = 0 & \text{on} \quad \mathsf{M}, \\ \mathcal{P}_\Sigma\left((\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) + \alpha u = 0 & \text{on} \quad \Sigma, \\ (u|\nu_\Sigma)_g = 0 & \text{on} \quad \Sigma, \end{cases}$$

satisfying the estimate

$$\lambda\|u\|_{L_q(\mathsf{M})} + \|u\|_{H_q^2(\mathsf{M})} + \|\mathsf{grad}\,\pi\|_{L_q(\mathsf{M})} \leq C\|f\|_{L_q(\mathsf{M})}. \quad \square$$

Proof of Theorem 2.1. The assertions (a) and (b) of the Theorem are contained in (4.1) and (4.2). $\square$

5. Existence and uniqueness of solutions

Based on the bounded H^∞-calculus property of A_N^{w} and A_N, the local well-posedness of (1.1) can be proved as in [30,32,37]. For the sake of completeness, we will nevertheless include a proof here.

By applying the Helmholtz projection $\mathbb{P}_H$ on $(1.1)_1$, one can readily see that the weak formulation of (1.1) is equivalent to the following abstract semilinear evolution equation

$$\begin{cases} \partial_t u + A_N^{\mathsf{w}} u = F^{\mathsf{w}}(u), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{5.1}$$

where for all $(u, v) \in H^1_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \times H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})$

$$\langle F^{\mathsf{w}}(u)|v\rangle_{\mathsf{M}} = (u \otimes u_\flat|\nabla v)_{\mathsf{M}},$$

where we used Lemma B.1 and the fact that $\nabla_u u = \mathsf{div}\,(u \otimes u)$ (which holds since $\mathsf{div}\, u = 0$). For notational brevity, we put

$$X_0^{\mathsf{w}} = X_{-1/2} \quad \text{and} \quad X_1^{\mathsf{w}} = X_{1/2}.$$

Due to (3.28),

$$X_\beta^{\mathsf{w}} := [X_0^{\mathsf{w}}, X_1^{\mathsf{w}}]_\beta = H_{q,\sigma}^{2\beta-1}(\mathsf{M}; T\mathsf{M}) \hookrightarrow L_{2q,\sigma}(\mathsf{M}; T\mathsf{M})$$

provided $2\beta - 1 \ge n/2q$. Then by taking $2\beta - 1 = n/2q$ and using Hölder's inequality, we infer that

$$\left|\langle F^{\mathsf{w}}(u)|v\rangle_{\mathsf{M}}\right| \le \|u\|_{L_{2q}(\mathsf{M})}^2 \|\nabla v\|_{L_{q'}(\mathsf{M})} \le C\|u\|_{X_\beta^{\mathsf{w}}}^2 \|v\|_{H_{q'}^1(\mathsf{M})}$$

and hence

$$\|F^{\mathsf{w}}(u)\|_{X_0^{\mathsf{w}}} \le C\|u\|_{X_\beta^{\mathsf{w}}}^2.$$

With $2\beta - 1 = n/2q$, which means $q \in (n/2, \infty)$ as $\beta < 1$, the critical weight μ_c^{w} and the corresponding critical space in the weak setting read as

$$X_{\gamma,\mu_c^{\mathsf{w}}}^{\mathsf{w}} = (X_0^{\mathsf{w}}, X_1^{\mathsf{w}})_{\mu_c^{\mathsf{w}}-1/p,p} = B_{qp,\sigma}^{n/q-1}(\mathsf{M}; T\mathsf{M}), \quad \mu_c^{\mathsf{w}} = 2\beta - 1 + \frac{1}{p} = \frac{1}{p} + \frac{n}{2q},$$

with $2/p + n/q \le 2$, see (3.28) and [32, Proposition 2.4 & Section 3.3] (for the Euclidean case).

Next, let us compute the critical spaces in the strong setting. To this end, we consider the following abstract evolution equation, which is equivalent to the strong formulation of (1.1)

$$\begin{cases} \partial_t u + A_N u = F(u) := -\mathbb{P}_H(\nabla_u u), \quad t > 0, \\ \qquad u(0) = u_0. \end{cases} \tag{5.2}$$

It follows from Hölder's inequality that

$$\|F(u)\|_{L_q(\mathsf{M})} \le C\|u\|_{L_{qr'}(\mathsf{M})}\|u\|_{H_{qr}^1(\mathsf{M})},$$

where $1/r + 1/r' = 1$. We choose $1 - \dfrac{n}{qr} = -\dfrac{n}{qr'}$, or equivalently $\dfrac{n}{qr} = \dfrac{1}{2}\left(1 + \dfrac{n}{q}\right)$, so that

$$H_{qr}^1(\mathsf{M}; T\mathsf{M}) \hookrightarrow L_{qr'}(\mathsf{M}; T\mathsf{M}).$$

Note that this choice is feasible if $q \in (1, n)$.

By interpolation theory and Sobolev's embedding theorem,

$$[X_0, X_1]_\beta \subset H_q^{2\beta}(\mathsf{M}; T\mathsf{M}) \hookrightarrow H_{qr}^1(\mathsf{M}; T\mathsf{M}),$$

provided

$$2\beta - \frac{n}{q} = 1 - \frac{n}{qr}, \quad \text{or equivalently} \quad \beta = \frac{1}{4}\left(\frac{n}{q} + 1\right).$$

In summary, we have shown that

$$\|F(u)\|_{L_q} \le C\|u\|_{X_\beta}^2, \quad u \in X_\beta.$$

The condition $\beta < 1$ requires $q > n/3$. Hence, for $q \in (n/3, n)$, the critical weight in the strong setting is given by

$$\mu_c = 2\beta - 1 + \frac{1}{p} = \frac{1}{2}\left(\frac{n}{q} - 1\right) + \frac{1}{p}, \quad \text{with} \quad \frac{2}{p} + \frac{n}{q} \le 3,$$

and the corresponding critical space in the strong setting is given by $X_{\gamma,\mu_c} := (X_0, X_1)_{\mu_c - 1/p, p}$, where

$$X_{\gamma,\mu_c} = B^{n/q-1}_{qp,\sigma,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) \text{ in case } n/q - 1 \ne 1 + 1/q, \tag{5.3}$$

see Proposition C.4.

The above discussions give rise to the following theorem concerning the local well-posedness of (5.1), respectively (1.1).

Theorem 5.1. (a) *Let $p \in (1,\infty)$ and $q \in (n/2,\infty)$ such that $\frac{2}{p} + \frac{n}{q} \le 2$. Then for any initial value $u_0 \in B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})$, there exists a unique weak solution*

$$u \in H^1_{p,\mu_c^{\mathsf{w}}}((0,t_+); H^{-1}_{q,\sigma}(\mathsf{M}; T\mathsf{M})) \cap L_{p,\mu_c^{\mathsf{w}}}((0,t_+); H^1_{q,\sigma}(\mathsf{M}; T\mathsf{M}))$$

of (5.1) for some $t_+ = t_+(u_0) > 0$ with $\mu_c^{\mathsf{w}} = 1/p + n/2q$. The solution exists on a maximal time interval $[0, t_{\max}(u_0))$ and depends continuously on u_0. Moreover,

$$u \in C([0,t_{\max}); B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})) \cap C((0,t_{\max}); B^{1-2/p}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})).$$

If, in addition, $q \ge n$, the solution u satisfies

$$u \in H^1_{p,loc}((0,t_{\max}); L_{q,\sigma}(\mathsf{M}; T\mathsf{M})) \cap L_{p,loc}((0,t_{\max}); H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})). \tag{5.4}$$

Hence, any solution regularizes instantaneously and becomes a strong solution in case $q \ge n$.

(b) *If $p \in (1,\infty)$ and $q \in (n/3, n)$ with $\frac{2}{p} + \frac{n}{q} \le 3$, then for any initial value $u_0 \in X_{\gamma,\mu_c}$, see (5.3) for a characterization, there exists a unique strong solution*

$$u \in H^1_{p,\mu_c}((0,t_+); L_{q,\sigma}(\mathsf{M}; T\mathsf{M})) \cap L_{p,\mu_c}((0,t_+); H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}))$$

of (5.2) for some $t_+ = t_+(u_0) > 0$ with $\mu_c = 1/p + n/2q - 1/2$. The solution exists on a maximal time interval $[0, t_{\max}(u_0))$ and depends continuously on u_0. Moreover,

$$u \in C([0,t_{\max}); B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})) \cap C((0,t_{\max}); B^{2-2/p}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})).$$

Proof. Because of (4.1) and (4.2), the local existence and uniqueness of a solution is an immediate consequence of [31, Theorem 1.2], see also [29, Theorem 2.1].

It remains to show the additional regularity property (5.4). Suppose $q \geq n$ and choose $r = 2p$. As $r > p$, we have

$$B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M}) \hookrightarrow B^{n/q-1}_{qr,\sigma}(\mathsf{M}; T\mathsf{M}) = (X^{\mathsf{w}}_0, X^{\mathsf{w}}_1)_{\mu_r - 1/r, r} =: X^{\mathsf{w}}_{\gamma,\mu_r}, \quad \text{with } \mu_r = 1/r + n/2q.$$

Note that $\mu_r < 1$. We can now consider problem (5.1) with initial value $u_0 \in X^{\mathsf{w}}_{\gamma,\mu_r}$. By uniqueness and [29, Theorem 2.1], we conclude that the solution u regularizes and satisfies

$$u(t_0) \in B^{1-2/r}_{qr,\sigma}(\mathsf{M}; T\mathsf{M}) = (X^{\mathsf{w}}_0, X^{\mathsf{w}}_1)_{1-1/r,r} =: X^{\mathsf{w}}_{\gamma,1}$$

for any $t_0 \in (0, t_{\max})$. Choosing $\mu \in (1/p, 1/2 + 1/2p)$, we have the embedding

$$B^{1-2/r}_{qr,\sigma}(\mathsf{M}; T\mathsf{M}) \hookrightarrow B^{2\mu-2/p}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})$$

at our disposal. Next, we note that

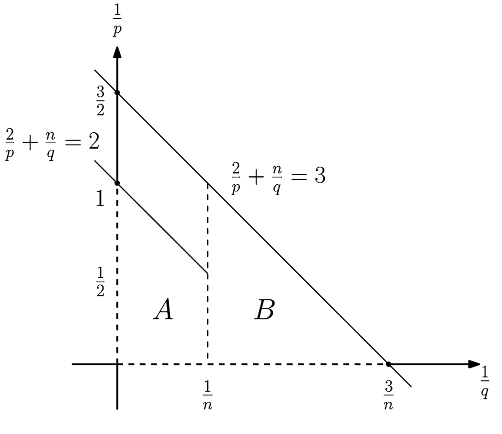
$$\|F(u)\|_{L_q(\mathsf{M})} \leq C\|u\|_{L_\infty(\mathsf{M})}\|u\|_{H^1_q(\mathsf{M})}$$

for all

$$u \in (X_0, X_1)_{\beta,p} \subset B^{2\beta}_{qp}(\mathsf{M}; T\mathsf{M}) \hookrightarrow L_\infty(\mathsf{M}; T\mathsf{M}) \cap H^1_q(\mathsf{M}; T\mathsf{M}),$$

provided $2\beta > 1$ and $q \geq n$. Now we can solve (5.2) with initial value $u(t_0) \in B^{2\mu-2/p}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})$ to obtain a strong solution by using [20, Theorem 2.1]. The asserted regularity (5.4) now follows from uniqueness of solutions. □

The following plot is helpful to illustrate the results in Theorem 5.1. When $(\frac{1}{q}, \frac{1}{p})$ is



- in region A we have weak solutions, which immediately regularize and become strong solutions;

- in region B, we have strong solutions.

The proof of Theorem 5.1 has an immediate byproduct.

Corollary 5.2. *Let $p \in (1, \infty)$, $q \in [n, \infty)$ and $\mu \in (1/p, 1]$. Then for any initial value $u_0 \in (X_0, X_1)_{\mu-1/p,p}$, where*

$$(X_0, X_1)_{\mu-1/p,p} = B^{2\mu-2/p}_{qp,\sigma,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) \quad \textit{in case} \quad 2\mu - 2/p \neq 1 + 1/q,$$

there exists a unique strong solution

$$u \in H^1_{p,\mu}((0, t_+); L_{q,\sigma}(\mathsf{M}; T\mathsf{M})) \cap L_{p,\mu}((0, t_+); H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}))$$

of (5.2) for some $t_+ = t_+(u_0) > 0$. The solution exists on a maximal time interval $[0, t_{\max}(u_0))$ and depends continuously on u_0. Moreover,

$$u \in C([0, t_{\max}); B^{2\mu-2/p}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})) \cap C((0, t_{\max}); B^{2-2/p}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})).$$

Remark 5.3. Concerning Theorem 5.1, two cases are of particular interest.

(i) Suppose that $n \geq 2$. Then for every $u_0 \in L_{n,\sigma}(\mathsf{M}; T\mathsf{M})$, (5.1) has a unique solution satisfying the regularity properties stated in Theorem 5.1(a) with $q = n$ for each fixed $p \geq n$. Therefore, Theorem 5.1 reproduces the celebrated results by Giga and Miyakawa [14] (obtained for no-slip boundary conditions) for Navier boundary conditions.

(ii) Suppose that $n = 2, 3$. Choosing $p = q = 2$ we can admit initial values $u_0 \in H^{n/2-1}_{2,\sigma}(\mathsf{M}; T\mathsf{M})$. This generalizes the celebrated results by Fujita and Kato [10,18].
In particular, if $n = 2$, for any $u_0 \in L_{2,\sigma}(\mathsf{M}; T\mathsf{M})$, (5.1) has a unique solution satisfying the regularity properties stated in Theorem 5.1(a) with $q = p = 2$. Moreover, the solution satisfies

$$u \in H^1_{p,loc}((0, t_{\max}); L_{q,\sigma}(\mathsf{M}; T\mathsf{M})) \cap L_{p,loc}((0, t_{\max}); H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}))$$

for any fixed $p, q > 1$.

Proof. The assertions can be shown by following the proofs of [30, Corollary 4.4 and Theorem 4.5] line by line. □

6. Large time behavior

6.1. Characterization of equilibria

We will begin the analysis of large time behavior by a characterization of the spectrum of A^{w}_N. Since $X_{1/2}$ is compactly embedded in $X_{-1/2}$, the spectrum $\sigma(A^{\mathsf{w}}_N)$ consists only of isolated eigenvalues and is independent of the choice of q. By Green's first identity, Lemma B.1, one obtains for all $(u, v) \in \mathsf{D}(A_{N,q}) \times H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})$

$$\begin{aligned}\langle A_N^{\mathsf{w}} u|v\rangle_{\mathsf{M}} &= (A_N u|v)_{\mathsf{M}} \\ &= 2\mu_s (D_u|D_v)_{\mathsf{M}} - \mu_s(\mathcal{P}_\Sigma(\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma|v)_\Sigma \\ &= 2\mu_s(D_u|D_v)_{\mathsf{M}} + \alpha\mu_s(\mathcal{P}_\Sigma u|\mathcal{P}_\Sigma v)_\Sigma \\ &= 2\mu_s(D_u|D_v)_{\mathsf{M}} + \alpha\mu_s(u|v)_\Sigma.\end{aligned} \tag{6.1}$$

By a density argument, one readily sees that (6.1) holds for all $(u, v) \in H^1_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \times H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})$. Suppose

$$A_N^{\mathsf{w}} u = \lambda u,$$

for some $\lambda \in \mathbb{C}$ and $u \in H^1_{2,\sigma}(\mathsf{M}; T\mathsf{M}_{\mathbb{C}})$, with $T\mathsf{M}_{\mathbb{C}} = T\mathsf{M} + iT\mathsf{M}$ denoting the complexified tangent bundle. (6.1) implies

$$\lambda\|u\|^2_{L_2(\mathsf{M})} = 2\mu_s\|D_u\|^2_{L_2(\mathsf{M})} + \alpha\mu_s\|u\|^2_{L_2(\Sigma)}, \tag{6.2}$$

and thus, $\lambda \geq 0$, i.e., $\sigma(A_N^{\mathsf{w}}) \subset [0, \infty)$. It follows that $\omega + A_N^{\mathsf{w}} \in \mathcal{S}(L_{q,\sigma}(\mathsf{M}; T\mathsf{M}))$ is invertible with spectral angle $\phi_{\omega + A_N^{\mathsf{w}}} < \pi/2$ for all $\omega > 0$. Applying one more time [28, Corollary 3.3.15] and taking advantage of (4.1) yields $\omega + A_N^{\mathsf{w}} \in H^\infty(X_{-1/2})$ with H^∞-angle $< \pi/2$, for all $\omega > 0$. Following the discussion in Section 4, it is not hard to see that the same holds true for A_N.

Moreover, (6.2) shows that $\mathsf{N}(A_N^{\mathsf{w}}) \subseteq \mathcal{E}_\alpha$, which is defined by

$$\mathcal{E}_\alpha = \begin{cases} \{u \in H^1_{2,\sigma}(\mathsf{M}; T\mathsf{M}) : D_u = 0 \text{ on } \mathsf{M} \quad \text{and} \quad u = 0 \text{ on } \Sigma\} & \text{if } \alpha > 0 \\ \{u \in H^1_{2,\sigma}(\mathsf{M}; T\mathsf{M}) : D_u = 0 \text{ on } \mathsf{M} \quad \text{and} \quad (u|\nu_\Sigma)_g = 0 \text{ on } \Sigma\} & \text{if } \alpha = 0. \end{cases}$$

Conversely, if $u \in \mathcal{E}_\alpha$, (6.1) implies that $A_N^{\mathsf{w}} u = 0$. Hence, $\mathsf{N}(A_N^{\mathsf{w}}) = \mathcal{E}_\alpha$.

Remark 6.1. Any element $u \in \mathcal{E}_\alpha$ is a Killing vector field on M, that is, it satisfies

$$(\nabla_v u|w)_g + (\nabla_w u|v)_g = 0, \quad \forall v, w \in C^\infty(\mathsf{M}; T\mathsf{M}). \tag{6.3}$$

Proposition 6.2.

$$\mathcal{E}_\alpha = \begin{cases} \{0\} & \textit{if } \alpha > 0 \\ \{u \in C^\infty(\mathsf{M}; T\mathsf{M}) : D_u = 0 \textit{ on } \mathsf{M} \quad \textit{and} \quad (u|\nu_\Sigma)_g = 0 \textit{ on } \Sigma\} & \textit{if } \alpha = 0. \end{cases}$$

Proof. Suppose $u \in H^1_{2,\sigma}(\mathsf{M}; T\mathsf{M})$ and $D_u = 0$. Then $(D_u)_\flat = 0$ as well. Let $u = u^k \frac{\partial}{\partial x^k}$ be a representation of u in local coordinates. Then, in local coordinates,

$$(D_u)_\flat = \left(g_{ki}u^k_{|j} + g_{kj}u^k_{|i}\right) dx^i \otimes dx^j =: \left(u_{i|j} + u_{j|i}\right) dx^i \otimes dx^j.$$

The relation $(D_u)_\flat = 0$ then reads

$$u_{i|j} + u_{j|i} = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - 2\Gamma^\ell_{ij} u_\ell = 0, \quad 1 \leq i, j \leq n.$$

Arguing as in the proof of [27, Lemma 3], we conclude that $u \in C^\infty(\mathsf{M}; T\mathsf{M})$.

Suppose now that $\alpha > 0$. Let $u \in \mathcal{E}_\alpha$. Since $u = 0$ on Σ, one can immediately infer that $\nabla_v u = 0$ on Σ for any $v \in C^\infty(\Sigma; T\Sigma)$. To show $\nabla_{\nu_\Sigma} u = 0$, we first observe that $(\nabla_{\nu_\Sigma} u|\nu_\Sigma)_g = 0$, as u is a Killing field, see (6.3). Next, we get for any $v \in C^\infty(\Sigma; T\Sigma)$

$$(\nabla_{\nu_\Sigma} u|v)_g = -(\nabla_v u|\nu_\Sigma)_g = 0,$$

where we use, once more, the property that u is a Killing field and $\nabla_v u = 0$. This yields $\nabla_v u = 0$ for each vector field $v \in C^\infty(\Sigma; T\mathsf{M})$, and hence $\nabla u = 0$ on Σ. It then follows from [26, Chapter 7, Proposition 28] that $u = 0$. □

Remark 6.3. Although, in general, $\mathcal{E}_0$ is non-trivial, $\mathcal{E}_0 = \{0\}$ holds under some specific conditions, for instance in case

(i) $\mathsf{Ric}^\sharp < 0$ and $L_\Sigma \geq 0$, or
(ii) $\mathsf{Ric}^\sharp \leq 0$ and $L_\Sigma > 0$,

see the proof of Lemma B.3, and also [41,42].

Proposition 6.4. *$\mathcal{E}_\alpha$ is the set of equilibria of (5.1), respectively (5.2). That is, the set of equilibria of (5.1), respectively (5.2), are exactly the Killing vector fields as characterized in Proposition 6.2.*

Proof. Assume that u_* is an equilibrium of (5.1), i.e., $u \in \mathsf{D}(A_N)$ and

$$A_N u_* = F(u_*).$$

Using the metric property of $(\cdot|\cdot)_g$ and the fact that $(u_*|\nu_\Sigma)_g = 0$, it follows from (B.1) that

$$(F(u_*)|u_*)_{\mathsf{M}} = -(\nabla_{u_*} u_*|u_*)_{\mathsf{M}} = -\frac{1}{2}\int_{\mathsf{M}} \nabla_{u_*}|u_*|_g^2 \, d\mu_g = 0.$$

Following the computations in (6.1), we get

$$0 = (F(u_*)|u_*)_{\mathsf{M}} = (A_N^{\mathsf{w}}(u_*)|u_*)_{\mathsf{M}} = 2\mu_s \|D_{u_*}\|_{L_2(\mathsf{M})}^2 + \alpha\mu_s \|u_*\|_{L_2(\Sigma)}^2.$$

Therefore, $u_* \in \mathcal{E}_\alpha$.

Conversely, if $u_* \in \mathcal{E}_\alpha$, then it is clear that $A_N u_* = 0$. On the other hand, by the definition of $\mathbb{P}_H$, one can show that

$$\mathbb{P}_H \nabla_{u_*} u_* = \mathbb{P}_H\left((\nabla u_*)u_*\right) = -\mathbb{P}_H\left([\nabla u_*]^{\mathsf{T}} u_*\right) = -\frac{1}{2}\mathbb{P}_H\left(\mathsf{grad}\,|u_*|_g^2\right) = 0.$$

This shows that u_* is an equilibrium of (5.2). □

With the convention that $H^0_{2,\sigma}(\mathsf{M}; T\mathsf{M}) := L_{2,\sigma}(\mathsf{M}; T\mathsf{M})$, we put

$$V_2^j = \{u \in H^j_{2,\sigma}(\mathsf{M}; T\mathsf{M}) : (u|v)_{\mathsf{M}} = 0 \quad \forall v \in \mathcal{E}_\alpha\}, \quad j = 0, 1. \tag{6.4}$$

In [27, Lemma 6], it was shown that $\mathcal{E}_\alpha$ is a finite dimensional space. Thus, V_2^j is a closed subspace of $H^j_{2,\sigma}(\mathsf{M}; T\mathsf{M})$ and

$$H^j_{2,\sigma}(\mathsf{M}; T\mathsf{M}) = \mathcal{E}_\alpha \oplus V_2^j, \quad j = 0, 1, \tag{6.5}$$

by a similar argument to [37, Remark 4.10 (a)].

6.2. *Global existence and convergence for 2D*

In this subsection, we consider the case $n = 2$. Given any $u_* \in \mathcal{E}_\alpha$, we consider the evolution equation

$$\begin{cases} \partial_t u + A_N u = G_*(u) := -\mathbb{P}_H(\nabla_u u + \nabla_u u_* + \nabla_{u_*} u), \quad t > 0, \\ \qquad u(0) = u_0 \end{cases} \tag{6.6}$$

and its weak counterpart

$$\begin{cases} \partial_t u + A^{\mathsf{w}}_N u = G^{\mathsf{w}}_*(u), \quad t > 0, \\ \qquad u(0) = u_0, \end{cases} \tag{6.7}$$

where

$$\langle G^{\mathsf{w}}_*(u)|\phi\rangle_{\mathsf{M}} = (u \otimes u_\flat|\nabla\phi)_{\mathsf{M}} + (u_* \otimes u_\flat|\nabla\phi)_{\mathsf{M}} + (u \otimes (u_*)_\flat|\nabla\phi)_{\mathsf{M}}, \quad \phi \in \dot{H}^1_{q',\sigma}(\mathsf{M}; T\mathsf{M}).$$

Note that, by choosing $p = q = 2$, the critical weight is $\mu^{\mathsf{w}}_c = 1$ and the corresponding critical trace space is

$$X^{\mathsf{w}}_{\gamma,\mu^{\mathsf{w}}_c} = (X_{-1/2}, X_{1/2})_{1/2,2} = [X_{-1/2}, X_{1/2}]_{1/2} = L_{2,\sigma}(\mathsf{M}; T\mathsf{M}),$$

see (3.28). Hence, by applying a similar argument to Remark 5.3 (ii), one can immediately infer that for every $u_0 \in L_{2,\sigma}(\mathsf{M}; T\mathsf{M})$, (6.7) has a unique solution

$$u \in H^1_2((0, t_+); H^{-1}_{2,\sigma}(\mathsf{M}; T\mathsf{M})) \cap L_2((0, t_+); H^1_{2,\sigma}(\mathsf{M}; T\mathsf{M}))$$

for some $t_+ = t_+(u_0) > 0$. The solution exists on a maximal time interval $[0, t_{\max}(u_0))$. In addition, it holds that

$$u \in H^1_{p,loc}((0, t_{\max}); L_{q,\sigma}(\mathsf{M}; T\mathsf{M})) \cap L_{p,loc}((0, t_{\max}); H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})) \tag{6.8}$$

for any fixed $p, q \in (1, \infty)$, and u also solves (6.6).

Next we show that any solution of (6.7) with initial value $u_0 \in L_{2,\sigma}(\mathsf{M})$ that is orthogonal to $\mathcal{E}_\alpha$ remains orthogonal for all later times. Moreover, we establish an energy estimate for such solutions.

Lemma 6.5. *Assume that $n = 2$. Given $u_0 \in V_2^0$, let u be the unique solution of (6.7). Then*

(a) $u(t) \in V_2^1$ for all $t \in (0, t_{\max}(u_0))$;
(b) there exists a constant $C > 0$ such that

$$\|u(t)\|^2_{L_2(\mathsf{M})} + C\int_0^t \|u(s)\|^2_{H_2^1(\mathsf{M})}\,ds \leq \|u_0\|^2_{L_2(\mathsf{M})}, \quad t \in (0, t_{\max}(u_0)); \tag{6.9}$$

(c) $t_{\max}(u_0) = +\infty$. Moreover, there exists a constant $\beta > 0$ such that

$$\|u(t)\|_{L_2(\mathsf{M})} \leq e^{-\beta t}\|u_0\|_{L_2(\mathsf{M})}, \quad t \geq 0. \tag{6.10}$$

Proof. (a) Pick any $z \in \mathcal{E}_\alpha$.

In the sequel, we suppress the time variable and simply write u in lieu of $u(t)$. Following the computations in (6.1), we have

$$\langle A_N u|z\rangle_{\mathsf{M}} = 2\mu_s(D_u|D_z)_{\mathsf{M}} + \alpha\mu_s(u|z)_\Sigma = 0. \tag{6.11}$$

Moreover, it holds that

$$\langle G_*(u)|z\rangle_{\mathsf{M}} = (\nabla_u u|z)_{\mathsf{M}} + (\nabla_{u_*}u|z)_{\mathsf{M}} + (\nabla_u u_*|z)_{\mathsf{M}} = 0. \tag{6.12}$$

We note that in (6.11) and (6.12) we may use the 'strong' operators A_N and G_*, as solutions immediately regularize, see (6.8). To show (6.12), we employ the metric property to obtain

$$(\nabla_{u_*}u|z)_g + (\nabla_u u_*|z)_g = \nabla_{u_*}(u|z)_g + \nabla_u(u_*|z)_g - (u|\nabla_{u_*}z)_g - (u_*|\nabla_u z)_g.$$

Since z is a Killing vector field, we infer that

$$(u|\nabla_{u_*}z)_g + (u_*|\nabla_u z)_g = 0, \tag{6.13}$$

see (6.3). Meanwhile, Lemma B.1, implies

$$\int_{\mathsf{M}} \left[\nabla_{u_*}(u|z)_g + \nabla_u(u_*|z)_g\right] d\mu_g = 0.$$

Similar computations show that

$$(\nabla_u u|z)_{\mathsf{M}} = 0.$$

Combining (6.11) and (6.12) yields

$$0 = \langle \partial_t u(t)|z\rangle_{\mathsf{M}} = \partial_t(u(t)|z)_{\mathsf{M}}.$$

Hence $(u(t)|z)_{\mathsf{M}} = 0$ and $u(t) \in V_2^1$ for all $t \in (0, t_{\max}(u_0))$.

(b) Due to (6.8), u is a valid test function in (6.6). Multiplying $(6.6)_1$ by u and integrating over M yields

$$\frac{d}{dt}\|u(t)\|^2_{L_2(\mathsf{M})} = -4\mu_s\|D(u)\|^2_{L_2(\mathsf{M})} - 2\alpha\mu_s\|u\|^2_{L_2(\Sigma)} + 2(G_*(u)|u)_{\mathsf{M}}.$$

Following a similar computation as in Part (a) and using the fact that u_* is a Killing vector field, one can show that

$$(G_*(u)|u)_{\mathsf{M}} = 0.$$

We thus have

$$\frac{d}{dt}\|u(t)\|^2_{L_2(\mathsf{M})} = -4\mu_s\|D(u)\|^2_{L_2(\mathsf{M})} - 2\alpha\mu_s\|u\|^2_{L_2(\Sigma)} \le -C\|u\|^2_{H^1_2(\mathsf{M})}, \tag{6.14}$$

where the last step follows from Korn's inequality, cf. Lemma B.3. Integrating both sides with respect to time gives (6.9).

(c) Part (b) shows that

$$u \in L_2((0, t_{\max}(u_0)); H^1_{2,\sigma}(\mathsf{M}; T\mathsf{M})).$$

It follows from [29, Theorem 2.4] that $t_{\max}(u_0) = +\infty$. An immediate consequence of (6.14) is

$$\frac{d}{dt}\|u(t)\|^2_{L_2(\mathsf{M})} + C\|u\|^2_{L_2(\mathsf{M})} \le 0, \quad \forall t > 0.$$

Solving the above ordinary differential inequality gives (6.10). □

Now we are in a position to prove the main theorem of this subsection.

Theorem 6.6. *Let $n = 2$. Then for every $u_0 \in L_{2,\sigma}(\mathsf{M}; T\mathsf{M})$, the unique solution u to (5.1) with initial value u_0 exists globally and enjoys the regularity properties listed in Remark 5.3 (ii). Furthermore, for any fixed $q \in (1, \infty)$, u converges to the equilibrium $u_* := \mathcal{P}_{\mathcal{E}_\alpha} u_0$ in the topology of $H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})$ at an exponential rate as $t \to \infty$, where $\mathcal{P}_{\mathcal{E}_\alpha}$ denotes the orthogonal projection from $L_{2,\sigma}(\mathsf{M}; T\mathsf{M})$ onto $\mathcal{E}_\alpha$.*

Proof. In view of (6.5), we can decompose u_0 into $u_0 = u_* + v_0$ such that $v_0 \in V^0_2$. Let $v(t)$ be the (unique) solution to

$$\begin{cases} \partial_t v + A^{\mathsf{w}}_N v = G^{\mathsf{w}}_*(v), & t > 0, \\ \qquad v(0) = v_0. \end{cases}$$

By Lemma 6.5, v exists globally. Then it follows from Proposition 6.4 that

$$u(t) = u_* + v(t)$$

is the unique global solution of (5.1) with initial value u_0. As was proved in Lemma 6.5,

$$\|u(t)-u_*\|_{L_2(\mathsf{M})} = \|v(t)\|_{L_2(\mathsf{M})} \le e^{-\beta t}\|v_0\|_{L_2(\mathsf{M})} = e^{-\beta t}\|u_0-u_*\|_{L_2(\mathsf{M})}, \quad t>0,$$

for some $\beta>0$. The convergence in the stronger topology $H^2_{q,\sigma}(\mathsf{M};T\mathsf{M})$ can be proved in the same way as in [37, Theorem 4.9]. □

6.3. Stability near killing vector fields

In this subsection, we will establish the stability of solutions of (5.1), respectively (1.1), with initial values close to a Killing vector field for $n>2$.

For any fixed $u_*\in\mathcal{E}_\alpha$, the linearization of the operator $[u\mapsto (A^{\mathsf{W}}_N u - F^{\mathsf{W}}(u))]$ is given by the operator $A^{\mathsf{W}}_0 : X_{1/2}\to X_{-1/2}$, defined by

$$\langle A^{\mathsf{W}}_0 u|v\rangle_{\mathsf{M}} = \langle A^{\mathsf{W}}_N u|v\rangle_{\mathsf{M}} - (u\otimes (u_*)_\flat + u_*\otimes u_\flat|\nabla v)_{\mathsf{M}}$$

for all $(u,v)\in H^1_{q,\sigma}(\mathsf{M};T\mathsf{M})\times H^1_{q',\sigma}(\mathsf{M};T\mathsf{M})$. In other words, $A^{\mathsf{W}}_0 = A^{\mathsf{W}}_N + B$, where B is the linear operator from $L_{q,\sigma}(\mathsf{M};T\mathsf{M})$ to $X_{-1/2}$ defined by

$$\langle Bu|v\rangle_{\mathsf{M}} = -(u\otimes (u_*)_\flat + u_*\otimes u_\flat|\nabla v)_{\mathsf{M}}, \quad v\in H^1_{q',\sigma}(\mathsf{M};T\mathsf{M}).$$

Proposition 6.2 shows that $\mathcal{E}_\alpha\subset C^\infty(\mathsf{M})$. Direct computations yield

$$|\langle Bu|v\rangle_{\mathsf{M}}| \le C\|u\|_{L_q(\mathsf{M})}\|v\|_{H^1_{q'}(\mathsf{M})}.$$

Therefore, $B\in\mathcal{L}(L_{q,\sigma}(\mathsf{M};T\mathsf{M}),X_{-1/2})$. From [28, Corollary 3.3.15], we infer that for some sufficiently large $\omega_0>0$

$$\omega + A^{\mathsf{W}}_0 \in H^\infty(X_{-1/2}) \text{ with } H^\infty\text{-angle } <\pi/2 \quad \text{for all } \omega>\omega_0.$$

Let $A_0\in\mathcal{L}(\mathsf{D}(A_{N,q}),L_{q,\sigma}(\mathsf{M};T\mathsf{M}))$ be the operator defined by

$$A_0 u = 2\mu_s\mathbb{P}_H \operatorname{div} D(u) + \mathbb{P}_H\left(\nabla_u u_* + \nabla_{u_*}u\right).$$

By applying a similar argument to A_N, we can show that by possibly further increasing $\omega_0>0$

$$\omega + A_0 \in H^\infty(L_{q,\sigma}(\mathsf{M};T\mathsf{M})) \text{ with } H^\infty\text{-angle } <\pi/2 \quad \text{for all } \omega>\omega_0.$$

Since $X_{1/2}$ is compactly embedded in $X_{-1/2}$, the spectrum of A^{W}_0 consists only of isolated eigenvalues and is independent of the choice of q. Suppose

$$A^{\mathsf{W}}_0 u = \lambda u,$$

for some $\lambda\in\mathbb{C}$. Following the computations in (6.1), it is not difficult to check that

$$\begin{aligned}\operatorname{Re}\lambda\|u\|^2_{L_2(\mathsf{M})} &= \operatorname{Re}(\langle A^{\mathsf{W}}_0 u|\overline{u}\rangle_{\mathsf{M}})\\ &= 2\mu_s\|D_u\|^2_{L_2(\mathsf{M})} + \alpha\mu_s\|u\|^2_{L_2(\Sigma)} + \operatorname{Re}(Bu|\overline{u})_{\mathsf{M}}.\end{aligned} \tag{6.15}$$

We observe that for all $v \in H^1_{2,\sigma}(\mathsf{M}; T\mathsf{M}_{\mathbb{C}})$, with $T\mathsf{M}_{\mathbb{C}}$ denoting the complexified tangent bundle,

$$\begin{aligned}\mathrm{Re}(Bv|\overline{v})_g &= -\mathrm{Re}\,(v \otimes (u_*)_\flat + u_* \otimes v_\flat|\nabla\overline{v})_g = -\mathrm{Re}\,[(u_*|\nabla_v\overline{v})_g + (v|\nabla_{u_*}\overline{v})_g] \\ &= -\mathrm{Re}\,\nabla_v(u_*|\overline{v})_g - \frac{1}{2}\nabla_{u_*}|v|_g^2 + \mathrm{Re}\,(\nabla_v u_*|\overline{v})_g,\end{aligned}$$

where we used the metric property of $(\cdot|\cdot)_g$. It follows from (B.1) that

$$\int_{\mathsf{M}} \left(\mathrm{Re}\nabla_v(u_*|\overline{v})_g - \frac{1}{2}\nabla_{u_*}|v|_g^2\right) d\mu_g = 0.$$

Finally, the definition of Killing vector fields implies

$$\mathrm{Re}(\nabla_v u_*|\overline{v})_g = 0.$$

Therefore, $\mathrm{Re}(Bu|\overline{u})_{\mathsf{M}} = 0$ and this shows that $\mathrm{Re}\lambda \geq 0$. When $\mathrm{Re}\lambda = 0$, one can infer from (6.15) that $u \in \mathcal{E}_\alpha$. This implies $\mathsf{N}(A_0^{\mathsf{w}}) \subseteq \mathcal{E}_\alpha$.

Conversely, if $z \in \mathcal{E}_\alpha$, then for any $v \in H^1_{q',\sigma}(\mathsf{M}; T\mathsf{M})$, the above computations show

$$\langle A_0^{\mathsf{w}} z|v\rangle_{\mathsf{M}} = (\mathbb{P}_H(\nabla_z u_* + \nabla_{u_*} z)|v)_{\mathsf{M}}.$$

As $D_{u_*} = D_z = 0$, we obtain

$$\begin{aligned}\mathbb{P}_H(\nabla_z u_* + \nabla_{u_*} z) &= \mathbb{P}_H((\nabla u_*)z + (\nabla z)u_*) = -\mathbb{P}_H((\nabla u_*)^{\mathsf{T}} z + (\nabla z)^{\mathsf{T}} u_*) \\ &= -\mathbb{P}_H \mathsf{grad}\,(u_*|z)_g = 0\end{aligned}$$

in virtue of the definition of $\mathbb{P}_H$. This implies that $z \in \mathsf{N}(A_0^{\mathsf{w}})$. In summary, we conclude that

$$\mathsf{N}(A_0^{\mathsf{w}}) = \mathcal{E}_\alpha$$

and $\sigma(A_0^{\mathsf{w}}) \cap i\mathbb{R} = \{0\}$. Next, we will show that the eigenvalue 0 of A_0^{w} is semi-simple. Indeed, if

$$A_0^{\mathsf{w}} u = z \in \mathcal{E}_\alpha,$$

then it follows from similar computations as in (6.11) and (6.13) that

$$\begin{aligned}\|z\|^2_{L_2(\mathsf{M})} &= \langle A_0^{\mathsf{w}} u|z\rangle_{\mathsf{M}} \\ &= \langle A_N^{\mathsf{w}} u|z\rangle_{\mathsf{M}} - (\mathbb{P}_H(\nabla_u u^* + \nabla_{u^*} u)|z)_{\mathsf{M}} \\ &= 2\mu_s(D_u|D_z)_{\mathsf{M}} + \alpha\mu_s(u|z)_\Sigma - (\nabla_u u^*|z)_{\mathsf{M}} - (\nabla_{u^*} u|z)_{\mathsf{M}} = 0.\end{aligned}$$

This shows that $z = 0$ and thus, $\mathsf{N}(A_0^{\mathsf{w}}) = \mathsf{N}((A_0^{\mathsf{w}})^2)$. As $\mathcal{E}_\alpha$ is a linear space, we clearly have $T_{u_*}\mathcal{E}_\alpha = \mathsf{N}(A_0^{\mathsf{w}})$. From Proposition 6.4 and [28], we learn that A_N^{w} is normally stable. So we can apply [28, Theorem 5.3.1] to obtain the following theorem.

Theorem 6.7. *Suppose that $n > 2$, $p \in (1, \infty)$ and $q \in (n/2, \infty)$ such that $\frac{2}{p} + \frac{n}{q} \le 2$. Then for each $u_* \in \mathcal{E}_0$, there exists some $\delta = \delta(u_*) > 0$ such that the solution u of (5.1) with initial value $u_0 \in B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})$ satisfying*

$$\|u_0 - u_*\|_{B^{n/q-1}_{qp}} \le \delta$$

exists globally and converges at an exponential rate to some $z \in \mathcal{E}_0$

(i) in the topology of $B^{1-2/p}_{qp}(\mathsf{M}; T\mathsf{M})$,
(ii) in the topology of $B^{2-2/p}_{qp}(\mathsf{M}; T\mathsf{M})$ if $q \ge n$.

Proof. According to Theorem 5.1, problem (5.1) has for each $u_0 \in B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})$ a unique solution u in the regularity class asserted by the theorem. In order to show assertions (i) and (ii), we will employ [28, Theorem 5.3.1] for initial values in $(X^{\mathsf{w}}_0, X^{\mathsf{w}}_1)_{1-1/r,r}$ for the weak setting, or in $(X_0, X_1)_{1-1/r,r}$ for the strong setting, with r properly chosen.

(i) We will first show that any solution u to (5.1) with initial value u_0 close to u_* in $B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})$ will also be close to u_* in $(X^{\mathsf{w}}_0, X^{\mathsf{w}}_1)_{1-1/r,r}$, for any fixed positive (sufficiently small) time and appropriate $r > p$.

Suppose $r > p$. As in the proof of Theorem 5.1, we have

$$B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M}) \hookrightarrow B^{n/q-1}_{qr,\sigma}(\mathsf{M}; T\mathsf{M}) = (X^{\mathsf{w}}_0, X^{\mathsf{w}}_1)_{\mu_r-1/r,r} =: X^{\mathsf{w}}_{\gamma,\mu_r}, \tag{6.16}$$

where $\mu_r = 1/r + n/2q$. Moreover,

$$u(t_0) \in B^{1-2/r}_{qr,\sigma}(\mathsf{M}; T\mathsf{M}) = (X^{\mathsf{w}}_0, X^{\mathsf{w}}_1)_{1-1/r,r} =: X^{\mathsf{w}}_{\gamma,1}$$

for any fixed time $t_0 \in (0, t_{\max})$. Using Lipschitz continuity of solutions with respect to initial data and the regularization property, there exists a positive number t_0 and a constant $C(t_0)$ such that

$$\|u(t_0) - u_*\|_{B^{1-2/r}_{qr}} \le C(t_0)\|u_0 - u_*\|_{B^{n/q-1}_{qr}} \le C(t_0)\|u_0 - u_*\|_{B^{n/q-1}_{qp}}, \tag{6.17}$$

for any initial value u_0 sufficiently close to u_* in $B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})$. Indeed, as solutions to (5.1) depend Lipschitz continuously on the initial data, see [29, Theorem 1.2], there are numbers t_0 and $M > 0$ such that

$$\|u - u_*\|_{\mathbb{E}^{\mathsf{w}}_{1,\mu_r}(0,2t_0)} \le M\|u_0 - u_*\|_{X^{\mathsf{w}}_{\gamma,\mu_r}} \tag{6.18}$$

for any initial value u_0 sufficiently close to u_* in $X^{\mathsf{w}}_{\gamma,\mu_r}$. Here we have set

$$\mathbb{E}^{\mathsf{w}}_{1,\mu}(T_1, T_2) := H^1_{p,\mu}((T_1, T_2); X^{\mathsf{w}}_0) \cap L_{p,\mu}((T_1, T_2); X^{\mathsf{w}}_1),$$

for $0 \le T_1 < T_2 < \infty$. Since $\mathbb{E}^{\mathsf{w}}_{1,\mu}(t_0, 2t_0) \hookrightarrow \mathbb{E}^{\mathsf{w}}_{1,1}(t_0, 2t_0) \hookrightarrow BUC((t_0, 2t_0); X^{\mathsf{w}}_{\gamma,1})$ for any $\mu \in (1/p, 1]$, we obtain with (6.18)

$$\begin{aligned}\|u(t_0)-u_*\|_{X^{\mathsf{w}}_{\gamma,1}} &\le \sup_{t\in[t_0,2t_0]}\|u(t)-u_*\|_{X^{\mathsf{w}}_{\gamma,1}} \le C\|u-u_*\|_{\mathbb{E}^{\mathsf{w}}_{1,1}(t_0,2t_0)} \\ &\le Ct_0^{\mu_r-1}\|u-u_*\|_{\mathbb{E}^{\mathsf{w}}_{1,\mu_r}(t_0,2t_0)} \le C(t_0)\|u_0-u_*\|_{X^{\mathsf{w}}_{\gamma,\mu_r}}.\end{aligned} \tag{6.19}$$

The assertion in (6.17) follows now from (6.16) and (6.19).

Hence $\|u(t_0)-u_*\|_{B^{1-2/r}_{qr}}$ can be made as small as we wish by making $\|u_0-u_*\|_{B^{n/q-1}_{qp}}$ small. It follows from the embedding

$$B^{1-2/r}_{qr,\sigma}(\mathsf{M};T\mathsf{M}) \hookrightarrow L_{2q,\sigma}(\mathsf{M};T\mathsf{M}),$$

which holds true as $4/r+n/q<2$ (here we need $r>2p$), and the estimate

$$(u_1\otimes(u_2)_\flat|\nabla v)_{\mathsf{M}} \le \|u_1\|_{L_{2q}(\mathsf{M})}\|u_2\|_{L_{2q}(\mathsf{M})}\|v\|_{H^1_{q'}(\mathsf{M})}$$

that $F^{\mathsf{w}}\in C^1(B^{1-2/r}_{qr,\sigma}(\mathsf{M};T\mathsf{M}),X^{\mathsf{w}}_0)$. We can now deduce from [28, Theorem 5.3.1] that each solution with initial value u_0 satisfying $\|u_0-u_*\|_{B^{n/q-1}_{qp}}\le\delta$, with $\delta>0$ sufficiently small, exists globally and converges exponentially fast to some $z\in\mathcal{E}_0$ in the topology of $B^{1-2/r}_{qr}(\mathsf{M};T\mathsf{M})$. The embedding

$$B^{1-2/r}_{qr}(\mathsf{M};T\mathsf{M}) \hookrightarrow B^{1-2/p}_{qp}(\mathsf{M};T\mathsf{M}),$$

then yields the assertion in (i).

We note that by the embedding $B^{1-2/p}_{qp}(\mathsf{M};T\mathsf{M})\hookrightarrow B^{n/q-1}_{qp}(\mathsf{M};T\mathsf{M})$, solutions also converge in the topology of critical spaces.

(ii) The arguments in step (i) show that $u(t_0)\in B^{1-2/r}_{qr}(\mathsf{M};T\mathsf{M})$ and that (6.17) holds true for any $r>p$ and any fixed time $t_0\in(0,t_{\max})$. In the following, we assume $r>\max\{p,2\}$. We then have the embedding

$$B^{1-2/r}_{qr,\sigma}(\mathsf{M};T\mathsf{M}) \hookrightarrow B^{2\mu-2/r}_{qr,\sigma}(\mathsf{M};T\mathsf{M})$$

for any fixed $\mu\in(1/r,1/2]$. We can now consider problem (5.2) with initial value

$$u(t_0)\in(X_0,X_1)_{\mu-1/r,r}=B^{2\mu-2/r}_{qr,\sigma}(\mathsf{M};T\mathsf{M}).$$

By regularization and uniqueness, we have

$$u(t_0+t_1)\in(X_0,X_1)_{1-1/r.r}\hookrightarrow B^{2-2/r}_{qr}(\mathsf{M};T\mathsf{M}),$$

for any fixed time $t_1>0$ such that $t_0+t_1<t_{\max}$.

An analogous argument to (6.19), with X^{w}_j replaced by X_j, $j=1,2$, and μ_r replaced by μ, shows that $\|u(t_0+t_1)-u_*\|_{B^{2-2/r}_{qr}}$ can be made as small as we wish by choosing $\|u_0-u_*\|_{B^{n/q-1}_{qp}}$ small.

The condition $r>2$ and $q\ge n$ ensures

$$(X_0,X_1)_{1-1/r,r}\hookrightarrow B^{2-2/r}_{qr}(\mathsf{M};T\mathsf{M})\hookrightarrow H^1_q(\mathsf{M};T\mathsf{M})\cap L_\infty(\mathsf{M};T\mathsf{M}).$$

Since $\|F(u)\|_{L_q(\mathsf{M})} \leq C\|u\|_{L_\infty(\mathsf{M})}\|u\|_{H^1_q(\mathsf{M})}$ we conclude that $F \in C^1((X_0, X_1)_{1-1/r,r}, X_0)$.

Theorem 5.3.1 in [28] then implies that each solution with initial value close to u_* in $(X_0, X_1)_{1-1/r,r}$ exists globally and converges exponentially fast to some $z \in \mathcal{E}_0$ in the topology of $B^{2-2/r}_{qr}(\mathsf{M}; T\mathsf{M})$. By the previous steps and the embedding $B^{2-2/r}_{qr}(\mathsf{M}; T\mathsf{M}) \hookrightarrow B^{2-2/p}_{qp}(\mathsf{M}; T\mathsf{M})$, which hold for ant $r > p$, we obtain assertion (ii). $\square$

Remark 6.8. In case $q \geq n$, analogous arguments as in [37, Remarks 4.10] show that every global solution of (5.1), respectively (5.2), with initial value u_0 converges exponentially fast to an equilibrium, namely to $\mathcal{P}_{\mathcal{E}_0} u_0$, where $\mathcal{P}_{\mathcal{E}_0}$ is the projection onto the finite dimensional space $\mathcal{E}_0$. Hence $z = \mathcal{P}_{\mathcal{E}_0} u_0$ in Theorem 6.7 in the particular case $q \geq n$.

Corollary 6.9. *Suppose that $\mathcal{E}_\alpha = \{0\}$, $p \in (1, \infty)$ and $q \in (n/2, \infty)$ such that $\frac{2}{p} + \frac{n}{q} \leq 2$. Then there exists some $\delta > 0$ such that the assertions (i) and (ii) of Theorem 6.7 hold true with $z = 0$ for any initial value $u_0 \in B^{n/q-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M})$ satisfying $\|u_0\|_{B^{n/q-1}_{qp}} \leq \delta$.*

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Appendix A. Tensor bundles and the Levi-Civita connection

Let M be a compact, smooth, and oriented n-dimensional Riemannian manifold with boundary $\Sigma = \partial\mathsf{M}$ and let $(\cdot|\cdot)_g$ denote the Riemann metric on M. We will use the same notation for the (induced) Riemann metric on Σ.

Then $T\mathsf{M}$ and $T^*\mathsf{M}$ denote the tangent and the cotangent bundle of M, respectively, and $T^\sigma_\tau\mathsf{M} := T\mathsf{M}^{\otimes\sigma} \otimes T^*\mathsf{M}^{\otimes\tau}$ stands for the (σ, τ)-tensor bundle of M for $\sigma, \tau \in \mathbb{N}$. The notations $\Gamma(\mathsf{M}; T^\sigma_\tau\mathsf{M})$ and $\mathcal{T}^\sigma_\tau\mathsf{M}$ stand for the set of all sections of $T^\sigma_\tau\mathsf{M}$ and the $C^\infty(\mathsf{M})$-module of all smooth sections of $T^\sigma_\tau\mathsf{M}$, respectively. For abbreviation, we put $\mathbb{J}^\sigma := \{1, 2, \ldots, n\}^\sigma$, and $\mathbb{J}^\tau$ is defined alike.

Given local coordinates $\{x^1, \ldots, x^n\}$,

$$(i) := (i_1, \cdots, i_\sigma) \in \mathbb{J}^\sigma, \quad (j) := (j_1, \cdots, j_\tau) \in \mathbb{J}^\tau,$$

we set

$$\frac{\partial}{\partial x^{(i)}} := \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_\sigma}}, \quad dx^{(j)} := dx^{j_1} \otimes \cdots \otimes dx^{j_\tau}.$$

Suppose that $a \in \Gamma(\mathsf{M}; T^\sigma_\tau\mathsf{M})$ is a $\mathbb{K}$-valued, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, tensor bundle on M. In this appendix, for notational brevity, we denote both $T^\sigma_\tau\mathsf{M}$ and its complexification by $T^\sigma_\tau\mathsf{M}$. The local representation of a with respect to these coordinates is given by

$$a = a^{(i)}_{(j)} \frac{\partial}{\partial x^{(i)}} \otimes dx^{(j)}, \qquad \text{with } a^{(i)}_{(j)} : U_k \to \mathbb{K},$$

where $U_k \subset \mathsf{M}$ is a coordinate patch.

For $s \in \{1, \dots, \sigma\}$, $t \in \{1, \dots, \tau\}$ and $a \in \Gamma(\mathsf{M}; T^\sigma_\tau \mathsf{M})$, $\mathsf{C}^s_t(a) \in \Gamma(\mathsf{M}; T^{\sigma-1}_{\tau-1}\mathsf{M})$ denotes the contraction of a with respect to the (s,t)-position. This means that in a local representation of a,

$$a = a^{(i_1,\dots,i_s,\dots,i_\sigma)}_{(j_1,\dots,j_t,\dots,j_\tau)} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_s}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_\sigma}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_t} \otimes \cdots \otimes dx^{j_\tau},$$

the terms $\frac{\partial}{\partial x^{i_s}}$ and dx^{j_t} are deleted and $a^{(i_1,\dots,i_s,\dots,i_\sigma)}_{(j_1,\dots,j_t,\dots,j_\tau)}$ is replaced by $a^{(i_1,\dots,k,\dots,i_\sigma)}_{(j_1,\dots,k,\dots,j_\tau)}$, and the sum convention is used for k.

Any $S \in \Gamma(\mathsf{M}; T^1_1\mathsf{M})$ induces a linear map from $\Gamma(\mathsf{M}; T\mathsf{M})$ to $\Gamma(\mathsf{M}; T\mathsf{M})$ by virtue of

$$Su = (S^i_j \frac{\partial}{\partial x^i} \otimes dx^j)u = S^i_j u^j \frac{\partial}{\partial x^i}, \quad u = u^j \frac{\partial}{\partial x^j} \in \Gamma(\mathsf{M}; T\mathsf{M}).$$

The dual S^* of $S \in \Gamma(\mathsf{M}; T^1_1\mathsf{M})$ is a linear map from $\Gamma(\mathsf{M}; T^*\mathsf{M})$ to $\Gamma(\mathsf{M}; T^*\mathsf{M})$, defined by

$$S^*\alpha = (S^i_j dx^j \otimes \frac{\partial}{\partial x^i})\alpha = S^i_j \alpha_i dx^j, \quad \alpha = \alpha_i dx^i \in \Gamma(\mathsf{M}; T^*\mathsf{M}).$$

The adjoint S^T of $S \in \Gamma(\mathsf{M}; T^1_1\mathsf{M})$ is the linear map from $\Gamma(\mathsf{M}; T\mathsf{M})$ to $\Gamma(\mathsf{M}; T\mathsf{M})$ defined by $S^\mathsf{T} = g^\sharp S^* g_\flat$, or more precisely,

$$S^\mathsf{T} u = g^\sharp[S^*(g_\flat u)], \quad u \in \Gamma(\mathsf{M}; T\mathsf{M}). \tag{A.1}$$

It holds that $(Su|v)_g = (u|S^\mathsf{T} v)_g$ for tangent fields u, v. In local coordinates, $S^\mathsf{T} = g^{i\ell} S^m_\ell g_{jm} \frac{\partial}{\partial x^i} \otimes dx^j$.

For $a \in \Gamma(\mathsf{M}; T^\sigma_\tau\mathsf{M})$, $\tau \geq 1$, $a^\sharp \in \Gamma(\mathsf{M}; T^{\sigma+1}_{\tau-1}\mathsf{M})$ is defined by

$$a^\sharp := g^\sharp a := \mathsf{C}^{\sigma+2}_1(a \otimes g^*),$$

and for $a \in \Gamma(\mathsf{M}; T^\sigma_\tau\mathsf{M})$, $\sigma \geq 1$, $a_\flat \in \Gamma(\mathsf{M}; T^{\sigma-1}_{\tau+1}\mathsf{M})$ is defined by

$$a_\flat := g_\flat a := \mathsf{C}^\sigma_1(g \otimes a).$$

Let ∇ be the Levi-Civita connection on M. For $u \in C^1(\mathsf{M}; T\mathsf{M})$, the covariant derivative $\nabla u \in C(\mathsf{M}; T^1_1\mathsf{M})$ is given in local coordinates by

$$\nabla u = \nabla_j u \otimes dx^j = (\partial_j u^i + \Gamma^i_{jk} u^k) \frac{\partial}{\partial x^i} \otimes dx^j =: u^i_{|j} \frac{\partial}{\partial x^i} \otimes dx^j,$$

where $u = u^i \frac{\partial}{\partial x^i}$, $\nabla_j = \nabla_{\frac{\partial}{\partial x^j}}$, and Γ^i_{jk} are the Christoffel symbols. It follows that $\nabla u + [\nabla u]^\mathsf{T}$ is given in local coordinates by

$$\nabla u + [\nabla u]^\mathsf{T} = \big(u^i_{|j} + g^{i\ell} u^m_{|\ell} g_{jm}\big) \frac{\partial}{\partial x^i} \otimes dx^j$$

and

$$(\nabla u + [\nabla u]^{\mathsf{T}})^{\sharp} = \left(g^{jk} u^i_{|k} + g^{ik} u^j_{|k}\right) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

The extension of the Levi-Civita connection on $C^1(\mathsf{M}; T^\sigma_\tau \mathsf{M})$ is again denoted by $\nabla := \nabla_g$. For $a \in C^1(\mathsf{M}; T^\sigma_\tau \mathsf{M})$, $\nabla a \in C(\mathsf{M}; T^\sigma_{\tau+1}\mathsf{M})$ is given in local coordinates by $\nabla a = \nabla_j a \otimes dx^j$, and

$$\mathsf{div} : C^1(\mathsf{M}; T^\sigma_\tau \mathsf{M}) \to C(\mathsf{M}; T^{\sigma-1}_\tau \mathsf{M}), \quad \sigma \geq 1,$$

is the divergence operator, defined by $\mathsf{div}\, a = \mathsf{C}^\sigma_{\tau+1}(\nabla a)$. In particular,

$$\mathsf{div}\, u = u^i_{|i} \quad \text{for} \quad u = u^i \frac{\partial}{\partial x^i}, \qquad \mathsf{div}\, S = S^{ik}_{|k} \frac{\partial}{\partial x^i} \quad \text{for} \quad S = S^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

For a scalar function $\phi \in C^1(\mathsf{M}; \mathbb{K})$, the gradient vector $\mathsf{grad}\, \phi \in C(\mathsf{M}; T\mathsf{M})$ is defined by the relation

$$(\mathsf{grad}\, \phi | u)_g := \langle \nabla \phi, u \rangle_g = \nabla_u \phi, \quad u \in C(\mathsf{M}; T\mathsf{M}),$$

where $\nabla \phi \in C(\mathsf{M}; T^*\mathsf{M})$ is the covariant derivative of ϕ. In local coordinates, we have

$$(\mathsf{grad}\, \phi)^i = g^{ij} \partial_j \phi, \quad 1 \leq i \leq n.$$

For the curvature tensor $R(u, v)w := [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w$, with $u, v, w \in \Gamma(\mathsf{M}; T\mathsf{M})$, we use the convention (as in [26,35], for instance)

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k} = R^\ell_{ijk} \frac{\partial}{\partial x^\ell}.$$

The Ricci tensor $\mathsf{Ric} \in T^0_2 \mathsf{M}$ is then defined by $\mathsf{Ric}_{jk} = R^i_{ijk}$.

The generalized metric g^τ_σ on $T^\sigma_\tau \mathsf{M}$ is still written as $(\cdot|\cdot)_g$. In addition,

$$|\cdot|_g : C^\infty(\mathsf{M}; T^\sigma_\tau \mathsf{M}) \to C^\infty(\mathsf{M}), \;\; a \mapsto \sqrt{(a|a)_g}$$

is called the (vector bundle) *norm* induced by g.

Appendix B. Some analysis on manifolds

Lemma B.1. *Let* $1 < q < \infty$.

(a) *Suppose that* $u \in H^1_q(\mathsf{M}; T\mathsf{M})$ *and* $\phi \in H^1_{q'}(\mathsf{M})$. *Then*

$$\begin{aligned} \int_{\mathsf{M}} (\mathsf{div}\, u)\phi \, d\mu_g &= -\int_{\mathsf{M}} (u|\mathsf{grad}\, \phi)_g \, d\mu_g + \int_\Sigma (u|\nu_\Sigma)_g \phi \, d\sigma_g \\ &= -\int_{\mathsf{M}} \nabla_u \phi \, d\mu_g + \int_\Sigma (u|\nu_\Sigma)_g \phi \, d\sigma_g, \end{aligned} \tag{B.1}$$

where μ_g (σ_g, respectively) is the volume element induced by g or $g|_\Sigma$, respectively.

(b) *(Green's first identity). Suppose that $S \in H^1_q(\mathsf{M}; T^2_0\mathsf{M})$ and $v \in H^1_{q'}(\mathsf{M}; T\mathsf{M})$. Then*

$$(\operatorname{div} S|v)_{\mathsf{M}} = -(S_\flat|\nabla v)_{\mathsf{M}} + (S_\flat \nu_\Sigma|v)_\Sigma. \tag{B.2}$$

In particular,

(i) $((\Delta_{\mathsf{M}} + \mathsf{Ric}^\sharp)u|v)_{\mathsf{M}} = -2(D_u|D_v)_{\mathsf{M}} + 2(D_u\nu_\Sigma|v)_\Sigma$, *with*

$$(u, v) \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \times H^1_{q'}(\mathsf{M}; T\mathsf{M});$$

(ii) $(\Delta_{\mathsf{M}}u|v)_{\mathsf{M}} = -(\nabla u|\nabla v)_{\mathsf{M}} + (\nabla u\, \nu_\Sigma|v)_\Sigma$, *with*

$$(u, v) \in H^2_q(\mathsf{M}; T\mathsf{M}) \times H^1_{q'}(\mathsf{M}; T\mathsf{M});$$

(iii) $(\operatorname{div}(u \otimes u)|v)_{\mathsf{M}} = -(u \otimes u_\flat|\nabla v)_{\mathsf{M}} + (u|v)_\Sigma\, (u|\nu_\Sigma)_\Sigma$, *with*

$$(u, v) \in \Big(H^1_q(\mathsf{M}; T\mathsf{M}) \cap L_\infty(\mathsf{M}; T\mathsf{M})\Big) \times H^1_{q'}(\mathsf{M}; T\mathsf{M}),$$

where $D_u = \frac{1}{2}(\nabla u + [\nabla u]^{\mathsf{T}})$ and $D_v = \frac{1}{2}(\nabla v + [\nabla v]^{\mathsf{T}})$.

Proof. (a) We first consider the case $u \in C^1(\mathsf{M}; T\mathsf{M})$ and $\phi \in C^1(\mathsf{M})$. The assertion follows from

$$\operatorname{div}(u\phi) = (\operatorname{div} u)\phi + (u|\operatorname{grad}\phi)_g = (\operatorname{div} u)\phi + \nabla_u\phi$$

and the divergence theorem on manifolds with boundary, cf. [21, Theorem 16.32]. In view of the fact that $\operatorname{div} \in \mathcal{L}(H^1_q(\mathsf{M}; T\mathsf{M}), L_q(\mathsf{M}; T\mathsf{M}))$, the assertion follows by a density argument.

(b) As in Part (a), it suffices to prove the assertion for $S \in C^1(\mathsf{M}; T^2_0\mathsf{M})$ and $v \in C^1(\mathsf{M}; T\mathsf{M})$. Then we have in local coordinates

$$S = S^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \quad v = v^i\frac{\partial}{\partial x^i}.$$

One readily verifies that

$$S^{\mathsf{T}}_\flat = g_{jk}S^{ji}\frac{\partial}{\partial x^i} \otimes dx^k, \quad S^{\mathsf{T}}_\flat v = g_{jk}S^{ji}v^k\frac{\partial}{\partial x^i}.$$

Direct computations show that in local coordinates

$$\operatorname{div}(S^{\mathsf{T}}_\flat v) = (g_{jk}S^{ji}v^k)_{|i} = (g_{jk}S^{ji})_{|i}v^k + g_{jk}S^{ji}v^k_{|i} = (\operatorname{div} S|v)_g + (S_\flat|\nabla v)_g.$$

By the divergence theorem on manifolds with boundary, cf. [21, Theorem 16.32],

$$\int_{\mathsf{M}} \operatorname{div}(S^{\mathsf{T}}_\flat v)\, d\mu_g = (S^{\mathsf{T}}_\flat v|\nu_\Sigma)_\Sigma = (S_\flat\nu_\Sigma|v)_\Sigma.$$

Hence (B.2) holds.

The assertion in (i) then follows by choosing $S = 2D(u)$ and noting that $\mathsf{div}\, 2D(u) = \Delta_{\mathsf{M}} u + \mathsf{Ric}^\sharp u$ as $\mathsf{div}\, u = 0$, see (1.2), and

$$(S_\flat|\nabla v)_g = (\nabla u + [\nabla u]^\mathsf{T}|\nabla v)_g = 2(D_u|D_v)_g.$$

(ii) follows by choosing $S = (\nabla u)^\sharp$ and noting that $S_\flat = \nabla u$ and $\mathsf{div}\,(\nabla u)^\sharp = \Delta_{\mathsf{M}} u$. Finally, the assertion in (iii) follows immediately by choosing $S = u \otimes u$. □

Lemma B.2. *Suppose $\phi \in H^3_q(\mathsf{M})$. Then*

$$\Delta_{\mathsf{M}}\, \mathsf{grad}\, \phi = \mathsf{grad}\, \Delta_B \phi + \mathsf{Ric}^\sharp \mathsf{grad}\, \phi.$$

Proof. Let $u = u^k \frac{\partial}{\partial x^k}$. Then $\Delta_{\mathsf{M}} u = g^{ij} u^k_{|i\,|j} \frac{\partial}{\partial x^k}$ and hence, $(\Delta_{\mathsf{M}} u)^k = g^{ij} u^k_{|i\,|j}$.

In case $u = \mathsf{grad}\, \phi = g^{kl} \phi_{|l} \frac{\partial}{\partial x^k}$ we obtain, employing the property that $g^{kl}_{|i} = 0$ for all $1 \le i, k, l \le n$,

$$\begin{aligned}
(\Delta_{\mathsf{M}}\, \mathsf{grad}\, \phi)^k &= g^{ij}(g^{kl}\phi_{|l})_{|i\,|j} = g^{ij} g^{kl} (\phi_{|l})_{|i\,|j} = g^{ij} g^{kl} (\phi_{|l\,|i})_{|j} = g^{ij} g^{kl} (\phi_{|i\,|l})_{|j} \\
&= g^{ij} g^{kl} (\phi_{|i})_{|l\,|j} = g^{kl} (g^{ij}(\phi_{|i})_{|l\,|j} = g^{kl} u^j_{|l\,|j} = g^{kl} u^j_{|j\,|l} + g^{kl}(u^j_{|l\,|j} - u^j_{|j\,|l}) \\
&= (\mathsf{grad}\, \mathsf{div}\, u)^k + g^{kl} \mathsf{Ric}_{lm} u^m = (\mathsf{grad}\, \mathsf{div}\, u + \mathsf{Ric}^\sharp u)^k \\
&= (\mathsf{grad}\, \Delta_B \phi + \mathsf{Ric}^\sharp\, \mathsf{grad}\, \phi)^k,
\end{aligned}$$

where we used the fact that $\phi_{|i\,|j} = \partial_j \partial_i \phi - \Gamma^k_{ji} \partial_k \phi = \phi_{|j\,|i}$ for scalar functions. □

Lemma B.3 *(Korn's inequality). There exists some constant $C > 0$ such that*

$$\|u\|_{H^1_2(\mathsf{M})} \le C \|D_u\|_{L_2(\mathsf{M})}, \quad u \in V^1_2, \tag{B.3}$$

where V^1_2 is defined in (6.4). In particular, if

(i) $\alpha > 0$, *or*
(ii) $\mathsf{Ric}^\sharp < 0$, $L_\Sigma \ge 0$ *and* $\alpha = 0$, *or*
(iii) $\mathsf{Ric}^\sharp \le 0$, $L_\Sigma > 0$ *and* $\alpha = 0$,

then (B.3) holds for all $u \in H^1_{2,\sigma}(\mathsf{M}; T\mathsf{M})$.

Proof. By combining assertions (i) and (ii) of Lemma B.1 (b), employing (2.8), and using compactness of M, we obtain

$$\begin{aligned}
2\|D_u\|^2_{L_2(\mathsf{M})} &= -(\Delta_M u|u)_{\mathsf{M}} - (\mathsf{Ric}^\sharp u|u)_{\mathsf{M}} + ((\nabla u + [\nabla u]^\mathsf{T})\nu_\Sigma|u)_\Sigma \\
&= \|\nabla u\|^2_{L_2(\mathsf{M})} - (\mathsf{Ric}^\sharp u|u)_{\mathsf{M}} + ((\nabla u + [\nabla u]^\mathsf{T})\nu_\Sigma|u)_\Sigma - (\nabla u \nu_\Sigma|u)_\Sigma \\
&= \|\nabla u\|^2_{L_2(\mathsf{M})} - (\mathsf{Ric}^\sharp u|u)_{\mathsf{M}} + (\mathcal{P}_\Sigma([\nabla u]^\mathsf{T} \nu_\Sigma)|u)_\Sigma \\
&= \|\nabla u\|^2_{L_2(\mathsf{M})} - (\mathsf{Ric}^\sharp u|u)_{\mathsf{M}} + (L_\Sigma u|u)_\Sigma \\
&\ge \|\nabla u\|^2_{L_2(\mathsf{M})} - c_1(\|u\|_{L_2(\mathsf{M})} + \|u\|_{L_2(\Sigma)})
\end{aligned} \tag{B.4}$$

for some constant c_1. Hence,

$$\|u\|_{H_2^1(\mathsf{M})} \le C\left(\|D_u\|_{L_2(\mathsf{M})} + \|u\|_{L_2(\mathsf{M})} + \|u\|_{L_2(\Sigma)}\right) \tag{B.5}$$

for some constant C. By trace theory, interpolation theory, see for instance [4, Theorem 10.1], and Young's inequality, we conclude that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\|u\|_{L_2(\Sigma)} \le \varepsilon \|u\|_{H_2^1(\mathsf{M})} + C(\varepsilon)\|u\|_{L_2(\mathsf{M})}.$$

Inequality (B.5) then becomes

$$\|u\|_{H_2^1(\mathsf{M})} \le C\left(\|D_u\|_{L_2(\mathsf{M})} + \|u\|_{L_2(\mathsf{M})}\right) \tag{B.6}$$

with a (possibly different) constant C.

The assertion in (B.3) then follows by a contradiction argument. Suppose (B.3) does not hold. Then there exists a sequence $\{u_n\}_{n=1}^\infty \subset V_2^1$ such that $\|u_n\|_{H_2^1(\mathsf{M})} = 1$ and

$$\|D_{u_n}\|_{L_2(\mathsf{M})} \to 0, \quad \text{as } n \to \infty.$$

Since V_2^1 is a closed subspace of $H_2^1(\mathsf{M}; T\mathsf{M})$, there exist a subsequence of $\{u_n\}_{n=1}^\infty$, not relabeled, and some $u \in V_2^1$ such that $u_n \to u$ in $L_{2,\sigma}(\mathsf{M}; T\mathsf{M})$ and $u_n \rightharpoonup u$ in $H_{2,\sigma}^1(\mathsf{M}; T\mathsf{M})$. It follows from (B.6) that $\{u_n\}_{n=1}^\infty$ is Cauchy in V_2^1 and thus, $u_n \to u$ in V_2^1. We can now infer that $\|D_{u_n} - D_u\|_{L_2(\mathsf{M})} \to 0$ as $n \to \infty$, and consequently, $u \in \mathcal{E}_\alpha$. Therefore, $u \in \mathcal{E}_\alpha \cap V_2^1 = \{0\}$. However, this contradicts the assumption that $\|u\|_{H_2^1(\mathsf{M})} = 1$. This completes the proof for (B.3).

Let us consider the set $\mathcal{E}_\alpha$ under conditions (i)-(iii). When $\alpha > 0$, it follows from Proposition 6.2 that $\mathcal{E}_\alpha = \{0\}$. Now we consider the case $\alpha = 0$. Let $u \in \mathcal{E}_0$ be given. Then $D_u = 0$. By the computations in (B.4), we have

$$0 = 2\|D_u\|^2_{L_2(\mathsf{M})} = \|\nabla u\|^2_{L_2(\mathsf{M})} - (\mathsf{Ric}^\sharp u|u)_{\mathsf{M}} + (L_\Sigma u|u)_\Sigma.$$

This shows that under assumptions (ii) or (iii), $u = 0$, and hence $\mathcal{E}_0 = \{0\}$. Therefore, in all three cases, we have $V_2^1 = H_{2,\sigma}^1(\mathsf{M}; T\mathsf{M})$. □

Remark B.4.
(a) In the Euclidean case, Korn's inequality for Navier boundary conditions was first proved in [39, Lemma 4].

(b) The estimate (B.5) remains valid for all $u \in H_2^1(\mathsf{M}; T\mathsf{M})$ satisfying $(u|\nu_\Sigma)_g = 0$, that is, without assuming that $\mathsf{div}\, u = 0$. Indeed, in this case, the assertion of Lemma B.1 (b) (i) reads

$$(\Delta_{\mathsf{M}} u + \mathsf{Ric}^\sharp u + \mathsf{grad}\,\mathsf{div}\, u|u)_{\mathsf{M}} = -2(D_u|D_u)_{\mathsf{M}} + 2(D_u \nu_\Sigma|u)_\Sigma, \quad u \in H_2^2(\mathsf{M}; T\mathsf{M}).$$

Using the relation $\mathsf{div}\,((\mathsf{div}\, u)u) = (\mathsf{div}\, u)^2 + (\mathsf{grad}\,\mathsf{div}\, u|u)_g$ and the assumption $(u|\nu_\Sigma)_g = 0$, we obtain by analogous arguments as above

$$
\begin{aligned}
2\|D_u\|^2_{L_2(\mathsf{M})} &= \|\nabla u\|^2_{L_2(\mathsf{M})} - (\mathsf{Ric}^\sharp u|u)_{\mathsf{M}} + (L_\Sigma u|u)_\Sigma + \|\mathsf{div}\, u\|^2_{L_2(\mathsf{M})} \\
&\geq \|\nabla u\|^2_{L_2(\mathsf{M})} - c_1(\|u\|_{L_2(\mathsf{M})} + \|u\|_{L_2(\Sigma)}).
\end{aligned}
$$

Hence the assertion (B.5) follows by a density argument.

In order to construct the Helmholtz projection on (M, g), we will need the following lemma, where we use the definition

$$
H^{-1}_q(\mathsf{M}) := \left(H^1_{q'}(\mathsf{M})\right)' \quad \text{and} \quad W^{-1/q}_q(\Sigma) := \left(W^{1/q}_{q'}(\Sigma)\right)', \quad 1/q + 1/q' = 1.
$$

We note that our definition of $H^{-1}_q(\mathsf{M})$ differs from the usual definition used in the literature. This abuse of notation allows for a more streamlined presentation of the results in the following two Lemmas.

Lemma B.5. *Let $q \in (1, \infty)$ and $k \in \{-1, 0, 1\}$. Then the Poisson problem*

$$
\begin{cases}
\Delta_B \phi = f & \text{on} \quad \mathsf{M}, \\
(\mathsf{grad}\, \phi|\nu_\Sigma)_g = h & \text{on} \quad \Sigma
\end{cases}
\tag{B.7}
$$

has a unique (up to a constant) solution $\phi \in H^{k+2}_q(\mathsf{M})$ for each $f \in H^k_q(\mathsf{M})$ and $h \in W^{k+1-1/q}_q(\Sigma)$ satisfying the solvability condition

$$
\langle f|1\rangle_{\mathsf{M}} = \langle h|1\rangle_\Sigma. \tag{B.8}
$$

Furthermore,

$$
\|\mathsf{grad}\, \phi\|_{H^{k+1}_q(\mathsf{M})} \leq C\left(\|f\|_{H^k_q(\mathsf{M})} + \|h\|_{W^{k+1-1/q}_q(\Sigma)}\right) \tag{B.9}
$$

for some constant $C > 0$. In case $k = -1$, equation (B.7) is interpreted as

$$
(\mathsf{grad}\, \phi|\mathsf{grad}\, v)_{\mathsf{M}} = \langle \mathcal{M}h - f|v\rangle_{\mathsf{M}}, \quad v \in H^1_{q'}(\mathsf{M}),
$$

where $\mathcal{M} \in \mathcal{L}(W^{-1/q}_q(\Sigma), H^{-1}_q(\mathsf{M}))$ is the dual of the trace operator $\mathrm{tr}_\Sigma \in \mathcal{L}(H^1_{q'}(\mathsf{M}), W^{1-1/q'}_{q'}(\Sigma))$.

Proof. For $k \in \{0, 1\}$ and $u \in H^{k+2}_q(\mathsf{M})$, let $\mathcal{B}u := (\mathrm{tr}_\Sigma\, \mathsf{grad}\, u|\nu_\Sigma)_g$, where tr_Σ denotes the trace operator. Then $\mathcal{B} \in \mathcal{L}(H^{k+2}_q(\mathsf{M}), W^{k+1-1/q}_q(\Sigma))$. Moreover, let

$$
\mathcal{A}_k : \mathsf{D}(\mathcal{A}_k) \to H^k_q(\mathsf{M}),\ \mathsf{D}(\mathcal{A}_k) = \{u \in H^{k+2}_q(\mathsf{M}) : \mathcal{B}u = 0 \text{ on } \Sigma\},\ \mathcal{A}_k u := -\Delta_B u.
$$

Following a localization argument as in Section 3.1, one can show that there exists $\omega_0 \in \mathbb{R}$ such that for all $\omega > \omega_0$

$$\omega + \mathcal{A}_k \in \mathcal{L}is(\mathsf{D}(\mathcal{A}_k), H^k_q(\mathsf{M})).$$

Since the embedding $H^{k+2}_q(\mathsf{M}) \hookrightarrow H^k_q(\mathsf{M})$ is compact, the spectrum $\sigma(\mathcal{A}_k)$ consists solely of isolated eigenvalues with finite multiplicity and the spectrum does not depend on $q \in (1, \infty)$. Let $\lambda \in \sigma(\mathcal{A}_s)$ and consider the eigenvalue problem

$$\lambda u = \mathcal{A}_k u \quad \text{in } \mathsf{M}.$$

Multiplying the above equality by $\overline{u}$ and applying Lemma B.1 yields

$$\lambda \|u\|_{L_2(\mathsf{M})} = \|\mathsf{grad}\, u\|_{L^2(\mathsf{M})},$$

which implies $\sigma(\mathcal{A}_k) \subset [0, \infty)$. In particular, we have

$$\mathsf{N}(\mathcal{A}_k) = \{u \in \mathsf{D}(\mathcal{A}_k) : u \equiv \text{constant}\} = \mathbb{R}1_{\mathsf{M}},$$

where 1_{M} is the constant 1 function on M. Next, we will show that $\lambda = 0$ is in fact a semi-simple eigenvalue of $\mathcal{A}_k$. Assume that $u \in \mathsf{N}(\mathcal{A}_k^2)$ and let

$$\mathcal{A}_k u = \phi.$$

Since $\phi \in \mathsf{N}(\mathcal{A}_k)$, it follows that $\phi \equiv$ constant. Multiplying both sides of the equation above by ϕ and using Lemma B.1 results in

$$(\mathcal{A}_k u|\phi)_{\mathsf{M}} = (\mathsf{grad}\, u|\mathsf{grad}\, \phi)_{\mathsf{M}} = 0 = \|\phi\|^2_{L_2(\mathsf{M})},$$

which further yields $\phi = 0$. Therefore, $\mathsf{N}(\mathcal{A}_k^2) = \mathsf{N}(\mathcal{A}_k)$. The assertion is thus established. This further implies that

$$H^k_q(\mathsf{M}) = \mathsf{N}(\mathcal{A}_k) \oplus \mathsf{R}(\mathcal{A}_k) = \mathbb{R}1_{\mathsf{M}} \oplus \mathsf{R}(\mathcal{A}_k).$$

Put $Y_0 = L_q(\mathsf{M})$ and $Y_1 = \mathsf{D}(A_0)$, where $A_0 := \omega + \mathcal{A}_0$ for a fixed number $\omega > 0$. We note that it follows from $\sigma(\mathcal{A}_0) \subset [0, \infty)$ that $\omega + \mathcal{A}_0 \in \mathcal{L}is(\mathsf{D}(\mathcal{A}_0), L_q(\mathsf{M}))$ for any $\omega > 0$.

The pair (Y_0, A_0) generates an interpolation-extrapolation scale with respect to the complex interpolation functor. We recall that $Y_1 := D(A_0) = \{u \in H^2_q(\mathsf{M}) : \mathcal{B}u = 0\}$. Let $Y_0^\sharp = L_{q'}(\mathsf{M})$ and

$$A_0^\sharp := (A_0)' = \omega + \mathcal{A}_0, \quad Y_1^\sharp := D(A_0^\sharp) = \{u \in H^2_{q'}(\mathsf{M}) : \mathcal{B}u = 0\}.$$

Then $(Y_0^\sharp, A_0^\sharp)$ also generates an interpolation-extrapolation scale $(Y_\beta^\sharp, A_\beta^\sharp)$, $\beta \in \mathbb{R}$, the dual scale.

By [2, Theorem V.1.5.12], it holds that $(Y_\beta)' = Y^\sharp_{-\beta}$ and $(A_\beta)' = A^\sharp_{-\beta}$ for $\beta \in \mathbb{R}$. In particular, when $\beta = -1/2$,

$$\begin{aligned}
&\mathsf{D}(A_{-1/2}) = Y_{1/2} = [Y_0, Y_1]_{1/2} = H^1_q(\mathsf{M}; T\mathsf{M}),\\
&Y_{-1/2} = (Y^\sharp_{1/2})' = ([Y_0^\sharp, Y_1^\sharp]_{1/2})' = (H^1_{q'}(\mathsf{M}; T\mathsf{M}))' = H^{-1}_q(\mathsf{M}),
\end{aligned}$$

see Proposition C.6. We have

$$A_{-1/2} = \omega + \mathcal{A}_{-1/2} : H^1_q(\mathsf{M}) = Y_{1/2} \to Y_{-1/2} = H^{-1}_q(\mathsf{M}),$$

where $\mathcal{A}_{-1/2}$ is characterized by

$$\langle \mathcal{A}_{-1/2}\phi | v\rangle_{\mathsf{M}} = (\operatorname{grad}\phi | \operatorname{grad} v)_{\mathsf{M}}, \quad v \in H^1_{q'}(\mathsf{M}),$$

see (B.1), and satisfies $\mathsf{N}(\mathcal{A}_{-1/2}) = \mathbb{R}1_{\mathsf{M}}$. Moreover,

$$H^{-1}_q(\mathsf{M}) = \mathbb{R}1_{\mathsf{M}} \oplus \mathsf{R}(\mathcal{A}_{-1/2}).$$

Particularly, this implies that $\mathcal{A}_{-1/2} \in \mathcal{L}is(H^1_q(\mathsf{M}) \cap \mathsf{R}(\mathcal{A}_{-1/2}), \mathsf{R}(\mathcal{A}_{-1/2}))$. In addition, observe that

$$\mathsf{R}(\mathcal{A}_{-1/2}) = \{u \in H^{-1}_q(\mathsf{M}) : \langle u | 1_{\mathsf{M}}\rangle_{\mathsf{M}} = 0\}.$$

Since $\operatorname{tr}_\Sigma \in \mathcal{L}(H^1_{q'}(\mathsf{M}), W^{1-1/q'}_{q'}(\Sigma))$, its dual

$$\mathcal{M} := (\operatorname{tr}_\Sigma|_{H^1_{q'}(\mathsf{M})})' \in \mathcal{L}(W^{-1/q}_q(\Sigma), H^{-1}_q(\mathsf{M})) \tag{B.10}$$

is well-defined. An important observation is that ϕ is a weak solution of (B.7) in $H^1_q(\mathsf{M})$ iff

$$\mathcal{A}_{-1/2}\phi = \mathcal{M}h - f, \tag{B.11}$$

or equivalently,

$$(\operatorname{grad}\phi | \operatorname{grad} v)_{\mathsf{M}} = \langle \mathcal{M}h - f | v\rangle_{\mathsf{M}}, \quad v \in H^1_{q'}(\mathsf{M}).$$

Since $\mathcal{M}h - f \in H^{-1}_q(\mathsf{M})$, it suffices to show that $\mathcal{M}h - f \in \mathsf{R}(\mathcal{A}_{-1/2})$. Indeed, due to (B.8)

$$\langle \mathcal{M}h - f | 1\rangle_{\mathsf{M}} = \langle h | 1\rangle_\Sigma - \langle f | 1\rangle_{\mathsf{M}} = 0.$$

This implies that (B.11) has a unique (up to a constant) weak solution $\phi \in H^1_q(\mathsf{M})$. Estimate (B.9) in the case $k = -1$ follows from

$$\begin{aligned}\|\operatorname{grad}\phi\|_{L_q(\mathsf{M})} &\le \|\phi\|_{H^1_q(\mathsf{M})} \le C\|\mathcal{M}h - f\|_{H^{-1}_q(\mathsf{M})} \le C\big(\|f\|_{H^{-1}_q(\mathsf{M})} + \|\mathcal{M}h\|_{H^{-1}_q(\mathsf{M})}\big)\\ &\le C\big(\|f\|_{H^{-1}_q(\mathsf{M})} + \|h\|_{W^{-1/q}_q(\Sigma)}\big).\end{aligned}$$

When $f \in H^k_q(\mathsf{M})$ and $h \in W^{k+1-1/q}_q(\Sigma)$ with $k \in \{0, 1\}$, it follows from [4, Theorem 10.1] that $\mathcal{B}$ has a right inverse $\mathcal{N}_k \in \mathcal{L}(W^{k+1-1/q}_q(\Sigma), H^{k+2}_q(\mathsf{M}))$. Observe that ϕ is a strong solution of (B.7) iff $\psi = \phi - \mathcal{N}_k h$ is a strong solution of

$$\begin{cases} \mathcal{A}_k\psi = \Delta_B \mathcal{N}_k h - f & \text{on} \quad \mathsf{M}, \\ (\operatorname{grad}\psi|\nu_\Sigma)_g = 0 & \text{on} \quad \Sigma. \end{cases} \tag{B.12}$$

Since $h \in W_q^{k+1-1/q}(\mathsf{M})$, it is an easy task to verify that $\Delta_B \mathcal{N}_k h \in H_q^k(\mathsf{M})$. Condition (B.8) and Lemma B.1 imply that $\Delta_B \mathcal{N}_k h - f \in \mathsf{R}(\mathcal{A}_k)$.

Therefore, (B.12) has a unique (up to a constant) strong solution $\psi \in H_q^{k+2}(\mathsf{M})$. The remaining cases in (B.9) can be established in a similar way to $k = -1$. This completes the proof. □

Lemma B.6. *Let $q \in (1, \infty)$ and $k \in \{-1, 0, 1\}$. For every $u \in H_q^{k+1}(\mathsf{M}; T\mathsf{M})$, the elliptic boundary value problem*

$$\begin{cases} \Delta_B \phi = \operatorname{div} u & \text{on} \quad \mathsf{M}, \\ (\operatorname{grad}\phi|\nu_\Sigma)_g = (u|\nu_\Sigma)_g & \text{on} \quad \Sigma \end{cases} \tag{B.13}$$

has a unique (up to a constant) solution $\phi \in H_q^{k+2}(\mathsf{M})$. The solution satisfies

$$\|\operatorname{grad}\phi\|_{H_q^{k+1}(\mathsf{M})} \le C\|u\|_{H_q^{k+1}(\mathsf{M})} \tag{B.14}$$

for some constant $C > 0$. In case $k = -1$, equation (B.13) is interpreted as

$$(\operatorname{grad}\phi|\operatorname{grad}v)_{\mathsf{M}} = (u|\operatorname{grad}v)_{\mathsf{M}}, \quad v \in H_{q'}^1(\mathsf{M}), \tag{B.15}$$

while (B.15) is always satisfied for solutions of (B.13) in case $k = 0, 1$.

Therefore, the Helmholtz projection $\mathbb{P}_H \in \mathcal{L}(H_q^{k+1}(\mathsf{M}; T\mathsf{M}), H_{q,\sigma}^{k+1}(\mathsf{M}; T\mathsf{M}))$ is well-defined.

Proof. Suppose first that $k \in \{0, 1\}$ and let $\mathcal{C}u := (\operatorname{tr}_\Sigma u|\nu_\Sigma)_g$ for $u \in H_q^{k+1}(\mathsf{M}; T\mathsf{M})$. Then by Lemma B.1(a), the pair

$$(f, g) = (\operatorname{div} u, \mathcal{C}u)$$

satisfies the solvability condition (B.8). Moreover, we have $\operatorname{div} u \in H_q^k(\mathsf{M})$ and $(u|\nu_\Sigma)_g \in W_q^{k+1-1/q}(\mathsf{M})$. The latter follows from the trace theorem, cf. [4, Theorem 10.1]. Solvability of (B.13) in these two cases thus follows from Lemma B.5.

Suppose $\phi \in H_q^{k+2}(\mathsf{M})$ is a solution of (B.13). Employing Lemma B.1 twice, we obtain

$$\begin{aligned} (\operatorname{grad}\phi|\operatorname{grad}v)_{\mathsf{M}} &= -(\operatorname{div} u|v)_{\mathsf{M}} + ((\operatorname{grad}\phi|\nu_\Sigma)_g|\operatorname{tr}_\Sigma v)_\Sigma \\ &= -(\operatorname{div} u|v)_{\mathsf{M}} + (\mathcal{C}u|\operatorname{tr}_\Sigma v)_\Sigma = (u|\operatorname{grad}v)_{\mathsf{M}} \end{aligned} \tag{B.16}$$

for all $v \in H_{q'}^1(\mathsf{M})$, showing (B.15).

We now consider the case $k = -1$. Let $\mathcal{M}$ be as in (B.10). Employing the same computation as in (B.16), we obtain

$$(\operatorname{div} u|v)_{\mathsf{M}} = (\mathcal{C}u|\operatorname{tr}_\Sigma v)_\Sigma - (u|\operatorname{grad}v)_{\mathsf{M}} = \langle \mathcal{M}(\mathcal{C}u)|v\rangle_{\mathsf{M}} - (u|\operatorname{grad}v)_{\mathsf{M}},$$

for each $(u, v) \in H_q^1(\mathsf{M}; T\mathsf{M}) \times H_{q'}^1(\mathsf{M})$. Hence,

$$|\langle \mathcal{M}(\mathcal{C}u) - \mathsf{div}\, u|v\rangle_{\mathsf{M}}| = |(u|\mathsf{grad}\, v)_{\mathsf{M}}| \le \|u\|_{L_q(\mathsf{M})}\|v\|_{H^1_{q'}(\mathsf{M})},$$

which further implies

$$[u \mapsto (\mathcal{M}(\mathcal{C}u) - \mathsf{div}\, u)] \in \mathcal{L}(H^1_q(\mathsf{M}; T\mathsf{M}), H^{-1}_q(\mathsf{M})). \tag{B.17}$$

By the density of $H^1_q(\mathsf{M}; T\mathsf{M})$ in $L_q(\mathsf{M}; T\mathsf{M})$ and (B.17), the operator $[u \mapsto (\mathcal{M}(\mathcal{C}u) - \mathsf{div}\, u)]$ has a unique continuous extension in $\mathcal{L}(L_q(\mathsf{M}; T\mathsf{M}), H^{-1}_q(\mathsf{M}; T\mathsf{M}))$, denoted by $\mathcal{F}$. The extension satisfies

$$(u|\mathsf{grad}\, v)_{\mathsf{M}} = \langle \mathcal{F}u|v\rangle_{\mathsf{M}}, \quad (u, v) \in L_q(\mathsf{M}; T\mathsf{M}) \times H^1_{q'}(\mathsf{M}).$$

By (B.16) and a density argument, we have

$$(\mathsf{grad}\, \phi|\mathsf{grad}\, v)_{\mathsf{M}} = (u|\mathsf{grad}\, v)_{\mathsf{M}} = \langle \mathcal{F}u|v\rangle_{\mathsf{M}}, \quad (u, v) \in L_q(\mathsf{M}; T\mathsf{M}) \times H^1_{q'}(\mathsf{M}).$$

Hence, (B.13) can be interpreted as

$$\mathcal{A}_{-1/2}\phi = \mathcal{F}u.$$

By analogous arguments as in the proof of Lemma B.5, this problem has (up to constants) a unique solution, which satisfies (B.14), as $\|\mathcal{F}u\|_{H^{-1}_q(\mathsf{M})} \le c\|u\|_{L_q(\mathsf{M})}$. □

Appendix C. Interpolation spaces

As in Section 3.2, let $A_0 = \omega + A_N : X_1 := \mathsf{D}(A_N) \to X_0$, for some $\omega > 0$, with $X_0 = L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$ and

$$X_1 = \{u \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}) : \alpha u + \mathcal{P}_\Sigma\left((\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) = 0 \text{ on } \Sigma\}.$$

Recall that A_0 is invertible. By [2, Theorems V.1.5.1 and V.1.5.4], the pair (X_0, A_0) generates an interpolation-extrapolation scale (X_β, A_β), $\beta \in \mathbb{R}$, with respect to the complex interpolation functor. When $\beta \in (0, 1)$, A_β is the X_β-realization of A_0, where

$$X_\beta = [X_0, X_1]_\beta$$

in view of (4.2). Let $X_0^\sharp := (X_0)' = L_{q',\sigma}(\mathsf{M}; T\mathsf{M})$ and

$$\begin{aligned} A_0^\sharp &:= (A_0)' = (\omega + A_N)' = \omega - \mu_s \mathbb{P}_H(\Delta_{\mathsf{M}} + \mathsf{Ric}^\sharp), \\ \mathsf{D}(A_0^\sharp) = X_1^\sharp &:= \{u \in H^2_{q',\sigma}(\mathsf{M}; T\mathsf{M}) : \alpha u + \mathcal{P}_\Sigma\left((\nabla u + [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) = 0 \text{ on } \Sigma\}. \end{aligned}$$

Then $(X_0^\sharp, A_0^\sharp)$ generates an interpolation-extrapolation scale $(X_\beta^\sharp, A_\beta^\sharp)$, $\beta \in \mathbb{R}$, the dual scale.

In the following, we set

$$H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) = \{u \in H^2_q(\mathsf{M}; T\mathsf{M}) : \mathcal{B}u = 0 \text{ on } \Sigma\},$$

where $\mathcal{B}u = (\mathcal{B}_1 u, \mathcal{B}_2 u) := (\mathrm{tr}_\Sigma\,(\mathcal{P}_\Sigma(\nabla u \nu_\Sigma) + (\alpha + L_\Sigma)\mathcal{P}_\Sigma u)\,, \mathrm{tr}_\Sigma(u|\nu_\Sigma)_g)$. One readily verifies that

$$X_1 = H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) \cap L_{q,\sigma}(\mathsf{M}; T\mathsf{M}). \tag{C.1}$$

Indeed, given any $u \in X_1$, we immediately have $u \in H^2_q(\mathsf{M}; T\mathsf{M}) \cap L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$ and the boundary condition $\mathcal{B}_2 u = 0$ is automatically satisfied, see (2.2). In view of (2.8), it holds that on Σ

$$0 = \alpha u + 2\mathcal{P}_\Sigma(D_u \nu_\Sigma) = \mathcal{P}_\Sigma(\nabla u \nu_\Sigma) + (\alpha + L_\Sigma)\mathcal{P}_\Sigma u = \mathcal{B}_1 u.$$

Therefore, we conclude that $X_1 \subset H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) \cap L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$. The converse inclusion $H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) \cap L_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \subset X_1$ follows from

$$H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) \cap L_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \subset H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})$$

and (2.8).

In order to characterize the interpolation spaces $X_\beta = [X_0, X_1]_\beta$ and $X_{\beta,p} = (X_0, X_1)_{\beta,p}$ we first include two auxiliary results.

Lemma C.1. *Given $\theta \in (0, 1)$ and $p \in (1, \infty)$, let $(\cdot, \cdot)_\theta$ stand for either the complex interpolation functor $[\cdot, \cdot]_\theta$, or the real interpolation functor $(\cdot, \cdot)_{\theta,p}$, respectively. Then*

$$(L_{q,\sigma}(\mathsf{M}; T\mathsf{M}), H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}))_\theta \doteq (L_q(\mathsf{M}; T\mathsf{M}), H^2_q(\mathsf{M}; T\mathsf{M}))_\theta \cap L_{q,\sigma}(\mathsf{M}; T\mathsf{M}).$$

Proof. Let $\widetilde{\mathbb{P}}_H := \mathbb{P}_H|_{H^2_q(\mathsf{M}; T\mathsf{M})}$. Then Lemma B.6 implies

$$\widetilde{\mathbb{P}}_H \in \mathcal{L}(H^2_q(\mathsf{M}; T\mathsf{M}), H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})).$$

Moreover, $\widetilde{\mathbb{P}}^2_H = \widetilde{\mathbb{P}}_H$ and $\widetilde{\mathbb{P}}_H u = u$ for all $u \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})$. The assertion then follows from [40, Theorem 1.17.1.1]. □

Lemma C.2. *Given $\theta \in (0, 1)$ and $p \in (1, \infty)$, let $(\cdot, \cdot)_\theta$ stand for either the complex interpolation functor $[\cdot, \cdot]_\theta$ or the real interpolation functor $(\cdot, \cdot)_{\theta,p}$. Then*

$$(X_0, X_1)_\theta \doteq (L_q(\mathsf{M}; T\mathsf{M}), H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M}))_\theta \cap L_{q,\sigma}(\mathsf{M}; T\mathsf{M}).$$

Proof. Define

$$A_\mathcal{B} : \mathsf{D}(\Delta_\mathcal{B}) := H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) \to L_q(\mathsf{M}; T\mathsf{M})$$

by $A_\mathcal{B} u := -\mu_s(\Delta_\mathsf{M} + \mathsf{Ric}^\sharp)u$. It follows from analogous arguments as in Section 3.1 that there exists λ_0 such that for all $\lambda > \lambda_0$

$$\lambda + A_\mathcal{B} \in \mathcal{L}is(H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M}), L_q(\mathsf{M}; T\mathsf{M})).$$

It follows from Lemma 4.3 that there exists λ_0 such that for all $\lambda > \lambda_0$

$$\lambda + A_N \in \mathcal{L}is(X_1, X_0).$$

Then the assertion follows from a similar argument to [3, Lemma 3.2]. For the reader's convenience, we will nevertheless include a proof. Let

$$Q_1 := (\lambda + A_N)^{-1}\mathbb{P}_H(\lambda + A_{\mathcal{B}}).$$

Then $Q_1 \in \mathcal{L}(H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M}), X_1)$ and

$$\begin{aligned} Q_1^2 u &= (\lambda + A_N)^{-1}\mathbb{P}_H(\lambda + A_{\mathcal{B}})(\lambda + A_N)^{-1}\mathbb{P}_H(\lambda + A_{\mathcal{B}})u \\ &= (\lambda + A_N)^{-1}\mathbb{P}_H(\lambda + A_{\mathcal{B}})u = Q_1 u, \end{aligned}$$

where we have employed the relations $\mathbb{P}_H A_{\mathcal{B}}|_{X_1} = A_N$ and $\mathbb{P}_H^2 = \mathbb{P}_H$. This further implies $Q_1|_{X_1} = I_{X_1}$, and thus Q_1 is a bounded projection from $H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M})$ onto X_1. Now consider Q_1 as a closed densely defined operator from $L_q(\mathsf{M}; T\mathsf{M})$ to $L_{q,\sigma}(\mathsf{M}; T\mathsf{M})$ with domain $H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M})$ and denote this operator by Q. Let

$$\begin{aligned} &A^\sharp_{\mathcal{B}} : H^2_{q',\mathcal{B}}(\mathsf{M}; T\mathsf{M}) \to L_{q'}(\mathsf{M}; T\mathsf{M}), \quad && A^\sharp_{\mathcal{B}} := -\mu_s(\Delta_\mathsf{M} + \mathsf{Ric}^\sharp) \\ &A^\sharp_N : \mathsf{D}(A_{N,q'}) \to L_{q',\sigma}(\mathsf{M}; T\mathsf{M}), && A^\sharp_N := -\mu_s\mathbb{P}_H(\Delta_\mathsf{M} + \mathsf{Ric}^\sharp). \end{aligned}$$

Then

$$\begin{aligned} Q' &= (\lambda + A_{\mathcal{B}})'\mathbb{P}_H^{\mathsf{T}}[(\lambda + A_N)^{-1}]' \\ &= (\lambda + A^\sharp_{\mathcal{B}})(\lambda + A^\sharp_N)^{-1} \in \mathcal{L}(L_{q',\sigma}(\mathsf{M}; T\mathsf{M}), L_{q'}(\mathsf{M}; T\mathsf{M})), \end{aligned}$$

where $\mathbb{P}_H^{\mathsf{T}}$ is the dual operator of $\mathbb{P}_H \in \mathcal{L}(L_q(\mathsf{M}; T\mathsf{M}), L_{q,\sigma}(\mathsf{M}; T\mathsf{M}))$. We note that $\mathbb{P}_H^{\mathsf{T}}$ is indeed the embedding operator $i^\sharp : L_{q',\sigma}(\mathsf{M}; T\mathsf{M}) \to L_{q'}(\mathsf{M}; T\mathsf{M})$. Therefore, $Q'' \in \mathcal{L}(L_q(\mathsf{M}; T\mathsf{M}), L_{q,\sigma}(\mathsf{M}; T\mathsf{M}))$. Together with the inclusion $Q \subset Q''$ and the density of $H^2_{q,\mathcal{B}}(\mathsf{M}; T\mathsf{M})$ in $L_q(\mathsf{M}; T\mathsf{M})$, this implies that Q has a unique bounded extension $Q_0 \in \mathcal{L}(L_q(\mathsf{M}; T\mathsf{M}), L_{q,\sigma}(\mathsf{M}; T\mathsf{M}))$. It is easy to check that Q_0 is a projection and $Q_0|_{X_0} = I_{X_0}$. Then the assertion follows from [40, Theorem 1.17.1.1]. □

We are now ready to state the first main result of this section, providing a characterization of the complex interpolation spaces $X_\beta := [X_0, X_1]_\beta$.

Proposition C.3. *Let $\beta \in (0, 1) \setminus \{\frac{1}{2} + \frac{1}{2q}\}$. Then $X_\beta = H^{2\beta}_{q,\sigma,\mathcal{B}}(\mathsf{M}; T\mathsf{M})$, where*

$$H^{2\beta}_{q,\sigma,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) = \begin{cases} \{u \in H^{2\beta}_{q,\sigma}(\mathsf{M}; T\mathsf{M}) : \alpha u + 2\mathcal{P}_\Sigma(D_u \nu_\Sigma) = 0 \ \text{ on } \ \Sigma\}, & \frac{1}{2} + \frac{1}{2q} < \beta < 1, \\ H^{2\beta}_{q,\sigma}(\mathsf{M}; T\mathsf{M}), & 0 < \beta < \frac{1}{2} + \frac{1}{2q}. \end{cases}$$

Proof. We first observe that

$$\mathcal{B}_1 \in \mathcal{L}(W_q^s(\mathsf{M};T\mathsf{M}), W_q^{s-1-1/q}(\Sigma;T\Sigma)), \quad 1+1/q < s \le 2,$$
$$\mathcal{B}_2 \in \mathcal{L}(W_q^s(\mathsf{M};T\mathsf{M}), W_q^{s-1/q}(\Sigma)), \quad 1/q < s \le 2,$$

are normal boundary operators in the sense of [36, Definition 3.1], see also [5, Section VIII.2]. Then [36, Theorem 4.1] implies, see also [5, Theorem 2.4.8] for the case $\mathsf{M} = \mathbb{R}^n_+$,

$$[L_q(\mathsf{M};T\mathsf{M}), H^2_{q,\mathcal{B}}(\mathsf{M};T\mathsf{M})]_\beta =: H^{2\beta}_{q,\mathcal{B}}(\mathsf{M};T\mathsf{M}),$$

where

$$H^{2\beta}_{q,\mathcal{B}}(\mathsf{M};T\mathsf{M}) \doteq \begin{cases} \{u \in H^{2\beta}_q(\mathsf{M};T\mathsf{M}) : \mathcal{B}u = 0 \quad \text{on } \Sigma\}, & \frac{1}{2}+\frac{1}{2q} < \beta < 1, \\ \{u \in H^{2\beta}_q(\mathsf{M};T\mathsf{M}) : \mathcal{B}_2 u = 0 \quad \text{on } \Sigma\}, & \frac{1}{2q} < \beta < \frac{1}{2}+\frac{1}{2q}, \\ H^{2\beta}_q(\mathsf{M};T\mathsf{M}), & 0 < \beta < \frac{1}{2q}. \end{cases}$$

Lemma C.2 shows that for $\beta \in (0,1) \setminus \{\frac{1}{2q}, \frac{1}{2}+\frac{1}{2q}\}$

$$[X_0, X_1]_\beta \doteq H^{2\beta}_{q,\mathcal{B}}(\mathsf{M};T\mathsf{M}) \cap L_{q,\sigma}(\mathsf{M};T\mathsf{M}).$$

By a similar argument as in (C.1), we obtain

$$\begin{aligned} &H^{2\beta}_{q,\mathcal{B}}(\mathsf{M};T\mathsf{M}) \cap L_{q,\sigma}(\mathsf{M};T\mathsf{M}) \\ &= \begin{cases} \{u \in H^{2\beta}_{q,\sigma}(\mathsf{M};T\mathsf{M}) : \alpha u + 2\mathcal{P}_\Sigma(D_u \nu_\Sigma) = 0 \quad \text{on } \Sigma\}, & \beta \in (\frac{1}{2}+\frac{1}{2q}, 1), \\ H^{2\beta}_{q,\sigma}(\mathsf{M};T\mathsf{M}), & \beta \in (0, \frac{1}{2}+\frac{1}{2q}) \setminus \{\frac{1}{2q}\}. \end{cases} \end{aligned}$$

We will now pay attention to the particular case $\beta = \frac{1}{2q}$, which is currently excluded in the characterization above. We know that $X_{1/2} = H^1_{q,\sigma}(\mathsf{M};T\mathsf{M})$. It follows from the reiteration theorem that

$$[X_0, X_{1/2}]_\alpha = [X_0, [X_0, X_1]_{1/2}]_\alpha = [X_0, X_1]_{\frac{\alpha}{2}} = X_{\alpha/2}.$$

Taking $\alpha = 1/q$ and using Lemma C.1 and the reiteration theorem yields

$$\begin{aligned} X_{1/2q} &\doteq [X_0, X_{1/2}]_{1/q} = [X_0, [X_0, H^2_{q,\sigma}(\mathsf{M};T\mathsf{M})]_{1/2}]_{1/q} \\ &= [X_0, H^2_{q,\sigma}(\mathsf{M};T\mathsf{M})]_{1/2q} \doteq H^{1/q}_{q,\sigma}(\mathsf{M};T\mathsf{M}). \end{aligned}$$

This proves the assertion for the case $\beta = 1/2q$ and thus completes the proof. □

We obtain an analogous result for the real interpolation spaces $X_{\beta,p} := (X_0, X_1)_{\beta,p}$.

Proposition C.4. *Let $\beta \in (0,1) \setminus \{\frac{1}{2}+\frac{1}{2q}\}$ and $p \in (1,\infty)$. Then*

$$X_{\beta,p} = (X_0, X_1)_{\beta,p} = B^{2\beta}_{qp,\sigma,\mathcal{B}}(\mathsf{M}; T\mathsf{M}), \quad \textit{where}$$

$$B^{2\beta}_{qp,\sigma,\mathcal{B}}(\mathsf{M}; T\mathsf{M}) = \begin{cases} \{u \in B^{2\beta}_{qp,\sigma}(\mathsf{M}; T\mathsf{M}) : \alpha u + 2\mathcal{P}_\Sigma (D_u \nu_\Sigma) = 0 \ \textit{on} \ \Sigma\}, & \frac{1}{2}+\frac{1}{2q} < \beta < 1, \\ B^{2\beta}_{qp,\sigma}(\mathsf{M}; T\mathsf{M}), & 0 < \beta < \frac{1}{2}+\frac{1}{2q}. \end{cases}$$

Proof. The case $\beta \in (0,1) \setminus \{\frac{1}{2q}, \frac{1}{2}+\frac{1}{2q}\}$ can be obtained as in Proposition C.3, see for instance [15] or [5, Theorem 2.4.5] for the Euclidean case.

To treat the case $\beta = \frac{1}{2q}$, note that the reiteration theorem for the complex and real method implies

$$X_{1/2q,p} = (X_0, X_1)_{1/2q,p} = (X_0, [X_0, X_1]_{1/2})_{1/q,p} = (X_0, X_{1/2})_{1/q,p}.$$

We can further utilize Lemma C.1 to obtain

$$\begin{aligned} X_{1/2q,p} &= (X_0, X_{1/2})_{1/q,p} = (X_0, [X_0, H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M})]_{1/2})_{1/q,p} \\ &= (X_0, H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}))_{1/2q,p} = B^{1/q}_{qp,\sigma}(\mathsf{M}; T\mathsf{M}). \end{aligned}$$

This completes the proof. □

Using a duality argument, we can also characterize the interpolation spaces between $X_{-1/2}$ and $X_{1/2}$.

Proposition C.5. *Let $\beta \in (0,1)$ and $p \in (1,\infty)$. Then*

$$[X_{-1/2}, X_{1/2}]_\beta = H^{2\beta-1}_{q,\sigma}(\mathsf{M}; T\mathsf{M}) \quad \textit{and} \quad (X_{-1/2}, X_{1/2})_{\beta,p} = B^{2\beta-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M}),$$

where

$$H^{2\beta-1}_{q,\sigma}(\mathsf{M}; T\mathsf{M}) := \left(H^{1-2\beta}_{q',\sigma}(\mathsf{M}; T\mathsf{M})\right)', \quad B^{2\beta-1}_{qp,\sigma}(\mathsf{M}; T\mathsf{M}) := \left(B^{1-2\beta}_{q'p',\sigma}(\mathsf{M}; T\mathsf{M})\right)'$$

for $\beta \in (0, 1/2)$.

Similar results hold for spaces with other boundary conditions. For instance, let

$$\begin{aligned} &Z_0 = X_0, \qquad Z_1 = \{u \in H^2_{q,\sigma}(\mathsf{M}; T\mathsf{M}) : \mathcal{P}_\Sigma\left((\nabla u - [\nabla u]^{\mathsf{T}})\nu_\Sigma\right) = 0 \ \text{ on } \ \Sigma\}, \quad \text{or} \\ &Y_0 = L_q(\mathsf{M}), \quad Y_1 = \{\phi \in H^2_q(\mathsf{M}) : (\operatorname{grad}\phi | \nu_\Sigma)_g = 0 \ \text{ on } \ \Sigma\}. \end{aligned}$$

Then we have the following result.

Proposition C.6. *Let $\beta \in (0,1)$. Then the interpolation spaces $Z_\beta = [Z_0, Z_1]_\beta$ and $Y_\beta = [Y_0, Y_1]_\beta$ can be characterized as follows.*

$$Z_\beta = \begin{cases} \{u \in H^{2\beta}_{q,\sigma}(\mathsf{M}; T\mathsf{M}) : \mathcal{P}_\Sigma\big((\nabla u - [\nabla u]^\mathsf{T})\nu_\Sigma\big) = 0 \ \ on \ \ \Sigma\}, & \frac{1}{2} + \frac{1}{2q} < \beta < 1, \\ H^{2\beta}_{q,\sigma}(\mathsf{M}; T\mathsf{M}), & 0 < \beta < \frac{1}{2} + \frac{1}{2q}. \end{cases}$$

$$Y_\beta = \begin{cases} \{\phi \in H^{2\beta}_{q}(\mathsf{M}) : (\operatorname{grad}\phi | \nu_\Sigma)_g = 0 \ \ on \ \ \Sigma\}, & \frac{1}{2} + \frac{1}{2q} < \beta < 1, \\ H^{2\beta}_{q}(\mathsf{M}), & 0 < \beta < \frac{1}{2} + \frac{1}{2q}. \end{cases}$$

Appendix D. Sectorial operators and H^∞-calculus

In this part of the appendix, we will introduce several basic concepts concerning maximal L_p-regularity theory. The reader may refer to the treatises [2], [8] and [28] for more details of these concepts.

For $\theta \in (0, \pi]$, the open sector with angle 2θ is denoted by

$$\Sigma_\theta := \{\omega \in \mathbb{C} \setminus \{0\} : |\arg \omega| < \theta\}.$$

Definition D.1. Let X be a complex Banach space, and $\mathcal{A}$ be a densely defined closed linear operator in X with dense range. $\mathcal{A}$ is called sectorial if $\Sigma_\theta \subset \rho(-\mathcal{A})$ for some $\theta > 0$ and

$$\sup\{\|\mu(\mu + \mathcal{A})^{-1}\|_{\mathcal{L}(X)} : \mu \in \Sigma_\theta\} < \infty.$$

The class of sectorial operators in X is denoted by $\mathcal{S}(X)$. The spectral angle $\phi_\mathcal{A}$ of $\mathcal{A}$ is defined by

$$\phi_\mathcal{A} := \inf\{\phi : \ \Sigma_{\pi-\phi} \subset \rho(-\mathcal{A}), \ \sup_{\mu \in \Sigma_{\pi-\phi}} \|\mu(\mu + \mathcal{A})^{-1}\|_{\mathcal{L}(X)} < \infty\}.$$

Let $\phi \in (0, \pi]$. Define

$$H^\infty(\Sigma_\phi) := \big\{f : \Sigma_\phi \to \mathbb{C} : f \text{ is analytic and } \|f\|_\infty < \infty\big\}$$

and

$$\mathcal{H}_0(\Sigma_\phi) = \left\{ f \in H^\infty(\Sigma_\phi) : \exists s > 0, c > 0 \text{ s.t. } |f(z)| \le c\frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

Definition D.2. Suppose that $\mathcal{A} \in \mathcal{S}(X)$. Then $\mathcal{A}$ is said to admit a bounded H^∞-calculus if there are $\phi > \phi_\mathcal{A}$ and a constant K_ϕ such that

$$\|f(\mathcal{A})\|_{\mathcal{L}(X)} \le K_\phi \|f\|_\infty, \quad f \in \mathcal{H}_0(\Sigma_{\pi-\phi}). \tag{D.1}$$

Here

$$f(\mathcal{A}) := -\frac{1}{2\pi i}\int_\Gamma (\lambda + \mathcal{A})^{-1} f(\lambda)\, d\lambda, \quad \Gamma = \begin{cases} -te^{-i\theta} & \text{for } t < 0, \\ te^{i\theta} & \text{for } t \ge 0, \end{cases} \tag{D.2}$$

is a positively oriented contour for any $\theta \in (0, \pi - \phi)$. The class of such operators is denoted by $H^\infty(X)$. The H^∞-angle of $\mathcal{A}$ is defined by

$$\phi_{\mathcal{A}}^\infty := \inf\{\phi > \phi_{\mathcal{A}} : (\text{D.1}) \text{ holds}\}.$$

If an operator $\mathcal{A} \in H^\infty(X)$ with H^∞-angle $\phi_{\mathcal{A}}^\infty < \pi/2$ and X is of class UMD, then Condition (H3) in [31] is satisfied with the choices $X_0 = X$ and $X_1 = \mathsf{D}(\mathcal{A})$.

Data availability

No data was used for the research described in the article.

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