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Dirichlet boundary value correction using Lagrange multipliers

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Abstract

We propose a boundary value correction approach for cases when curved boundaries are approximated by straight lines (planes) and Lagrange multipliers are used to enforce Dirichlet boundary conditions. The approach allows for optimal order convergence for polynomial order up to 3. We show the relation to a Taylor series expansion approach previously used in the context of Nitsche's method and, in the case of *inf-sup* stable multiplier methods, prove a priori error estimates with explicit dependence on the meshsize and distance between the exact and approximate boundary.

1 Introduction

In this contribution we develop a modified Lagrange multiplier method based on the idea of boundary value correction originally proposed for standard finite element

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methods on an approximate domain in [1] and further developed in [2]. More recently boundary value correction have been developed for cut and immersed finite element methods [3–7]. Using the closest point mapping to the exact boundary, or an approximation thereof, the boundary condition on the exact boundary may be weakly enforced using multipliers on the boundary of the approximate domain. Of particular practical importance in this context is the fact that we may use a piecewise linear approximation of the boundary, which is very convenient from a computational point of view since the geometric computations are simple in this case and a piecewise linear distance function may be used to construct the discrete domain.

We first compare the formulation with the one using Nitsche's method introduced for cut finite element methods in [3] and show how this can be interpreted as an augmented Lagrangian formulation of the multiplier method, where the multiplier has been eliminated in the spirit of [8].

We then prove a priori error estimates in the energy and L^2 norms, in terms of the error in the boundary approximation and the meshsize. We obtain optimal order convergence for polynomial approximation up to order 3 of the solution.

Note that without boundary correction one typically requires $O(h^{p+1})$ accuracy in the L^∞ norm for the approximation of the domain, which leads to significantly more involved computations on the cut elements for higher order elements, see [9]. We present numerical results illustrating our theoretical findings.

The outline of the paper is as follows: In Sect. 2 we formulate the model problem and our method, in Sect. 3 we present our theoretical analysis, in Sect. 4 we discuss the choice of finite element spaces in cut finite element methods, in Sect. 5 we present the numerical results, and finally in Sect. 6 we include some concluding remarks.

2 Model problem and method

2.1 The domain

Let Ω be a domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{n}$. We let ϱ be the signed distance function, negative on the inside and positive on the outside, to $\partial\Omega$ and we let $U_\delta(\partial\Omega)$ be the tubular neighborhood $\{\boldsymbol{x} \in \mathbb{R}^d : |\varrho(\boldsymbol{x})| < \delta\}$ of $\partial\Omega$. Then there is a constant $\delta_0 > 0$ such that the closest point mapping $\boldsymbol{p}(\boldsymbol{x}) : U_{\delta_0}(\partial\Omega) \to \partial\Omega$ is well defined and we have the identity $\boldsymbol{p}(\boldsymbol{x}) = \boldsymbol{x} - \varrho(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{p}(\boldsymbol{x}))$. We assume that δ_0 is chosen small enough that $\boldsymbol{p}(\boldsymbol{x})$ is well defined. See [10, Sect. 14.6], for further details on distance functions.

2.2 The model problem

We consider the problem: find $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega \tag{2.1}$$
$$u = g \quad \text{on } \partial\Omega \tag{2.2}$$

where $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ are given data. It follows from the Lax–Milgram lemma that there exists a unique solution to this problem and we also have the elliptic regularity estimate

$$\|u\|_{H^{s+2}(\Omega)} \lesssim \|f\|_{H^s(\Omega)} + \|g\|_{H^{\frac{3}{2}+s}(\partial\Omega)}, \quad s \geq -1. \tag{2.3}$$

Here and below we use the notation $\lesssim$ to denote less or equal up to a constant.

Using a Lagrange multiplier to enforce the boundary condition we can write the weak form of (2.4)–(2.5) as: find $(u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\partial\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega + \int_{\partial\Omega} \lambda \, v \, ds = \int_\Omega f v \, d\Omega \quad \forall v \in H^1(\Omega) \tag{2.4}$$

$$\int_{\partial\Omega} u \, \mu \, ds = \int_{\partial\Omega} g \, \mu \, ds \quad \forall \mu \in H^{-1/2}(\partial\Omega) \tag{2.5}$$

2.3 The mesh and the discrete domain

Let $\{\mathcal{K}_h\}, h \in (0, h_0]$, be a family of quasiuniform partitions, with mesh parameter h, consisting of shape regular triangles or tetrahedra K, with diameter h_K, $h = \max_{K \in \mathcal{K}_h} h_K$ and $h_{min} = \min_{K \in \mathcal{K}_h} h_K$. The partitions induce discrete polygonal approximations $\Omega_h = \cup_{K \in \mathcal{K}_h} K, h \in (0, h_0]$, of Ω. We assume neither $\Omega_h \subset \Omega$ nor $\Omega \subset \Omega_h$, instead the accuracy with which Ω_h approximates Ω will be crucial. For each $\mathcal{K}_h$, $\partial\Omega_h$ is given by the trace mesh consisting of the set of faces in the elements $K \in \mathcal{K}_h$ that belong to only one element. To each Ω_h is associated a discrete unit normal $\boldsymbol{n}_h$ and a discrete signed distance $\varrho_h : \partial\Omega_h \to \mathbb{R}$, such that if $\boldsymbol{p}_h(\boldsymbol{x}, \varsigma) := \boldsymbol{x} + \varsigma \boldsymbol{n}_h(\boldsymbol{x})$ then $\boldsymbol{p}_h(\boldsymbol{x}, \varrho_h(\boldsymbol{x})) \in \partial\Omega$ for all $\boldsymbol{x} \in \partial\Omega_h$. We will also assume that $\boldsymbol{p}_h(\boldsymbol{x}, \varsigma) \in U_{\delta_0}(\Omega) := U_{\delta_0}(\partial\Omega) \cup \Omega$ for all $\boldsymbol{x} \in \partial\Omega_h$ and all ς between 0 and $\varrho_h(\boldsymbol{x})$. For conciseness we will drop the second argument of $\boldsymbol{p}_h$ below whenever it takes the value $\varrho_h(\boldsymbol{x})$, and thus we have the map $\partial\Omega_h \ni \boldsymbol{x} \mapsto \boldsymbol{p}_h(\boldsymbol{x}) \in \partial\Omega$. We make the following assumptions

$$\delta_h := \|\varrho_h\|_{L^\infty(\partial\Omega_h)} = O(h), \quad h \in (0, h_0] \tag{2.6}$$

where $O(\cdot)$ denotes the big ordo. We also assume that h_0 is small enough to guarantee that

$$\partial\Omega_h \subset U_{\delta_0}(\partial\Omega), \quad h \in (0, h_0] \tag{2.7}$$

and that there exists $M > 0$ such for any $\boldsymbol{y} \in U_{\delta_0}(\partial\Omega)$ the equation, find $\boldsymbol{x} \in \partial\Omega_h$ and $|\varsigma| \leq \delta_h$ such that

$$\boldsymbol{p}_h(\boldsymbol{x}, \varsigma) = \boldsymbol{y} \tag{2.8}$$

has a solution set $\mathcal{P}_h$ with

$$\text{card}(\mathcal{P}_h) \leq M \tag{2.9}$$

uniformly in h. The rationale of this assumption is to ensure that the image of $\boldsymbol{p}_h$ can not degenerate for vanishing h; for more information cf. [3].

We note that it follows from (2.6) that

$$\|\varrho\|_{L^\infty(\partial\Omega_h)} \leq \|\varrho_h\|_{L^\infty(\partial\Omega_h)} = O(h) \tag{2.10}$$

since $|\varrho_h(\boldsymbol{x})| \geq |\varrho(\boldsymbol{x})|$, $\boldsymbol{x} \in \partial\Omega_h$. For stability we need a bound on how much Ω_h can overshoot Ω, more precisely we assume that $\varrho_h \geq -C_{\partial\Omega} h_{min}$ for a $C_{\partial\Omega}$ small enough that will be determined by the analysis but essentially depends on the constants associated to the stability of the finite element pair used.

Since Ω_h and Ω differ we need to extend the data f and the solution u in a smooth fashion. To this end we recall [11, Sect. 2.3, Theorem 5] the stable extension operator $E : H^m(\Omega) \mapsto H^m(\mathbb{R}^d)$, $m \geq 0$ satisfying

$$\|Eu\|_{H^m(\mathbb{R}^d)} \lesssim \|Eu\|_{H^m(\Omega)}. \tag{2.11}$$

We will also denote an extended function by $v^E := Ev$.

2.4 The finite element method

2.4.1 Boundary value correction

The basic idea of the boundary value correction of [1] is to use a Taylor series at $\boldsymbol{x} \in \partial\Omega_h$ in the direction $\boldsymbol{n}_h$, and let this series represent $u_h|_{\partial\Omega}$. For and $\boldsymbol{x} \in \partial\Omega_h$ we may write

$$u \circ \boldsymbol{p}_h(\boldsymbol{x}) = u(\boldsymbol{x}) + \varrho_h(\boldsymbol{x})\boldsymbol{n}_h(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}) + T_2^\Delta(u)(\boldsymbol{x}), \quad \text{for } \boldsymbol{x} \in \partial\Omega_h, \tag{2.12}$$

with

$$T_2^\Delta(u)(\boldsymbol{x}) = \int_0^{\varrho_h} \int_0^s \partial_t^2 u^E(\boldsymbol{x} + t\boldsymbol{n}_h) \, dt \, ds.$$

Below we will drop the second argument of T_2^Δ. In the present work we will restrict the discussion to methods using first two terms of the right hand side to approximate $u \circ \boldsymbol{p}_h(\boldsymbol{x})$,

$$u \circ \boldsymbol{p}_h(\boldsymbol{x}) \approx u(\boldsymbol{x}) + \varrho_h(\boldsymbol{x})\boldsymbol{n}_h(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}),$$

which are the ones of most practical interest.

Choosing appropriate discrete spaces V_h and Λ_h for the approximation of u and λ, respectively (particular choices are considered in Sect. 5), we thus seek $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v \, d\Omega_h + \int_{\partial\Omega_h} \lambda_h \, v \, ds = \int_{\Omega_h} f^E v \, d\Omega_h \quad \forall v \in V_h \tag{2.13}$$

$$\int_{\partial\Omega_h} (u_h + \varrho_h \boldsymbol{n}_h \cdot \nabla u_h) \mu \, ds = \int_{\partial\Omega_h} \tilde{g} \, \mu \, ds \quad \forall \mu \in \Lambda_h \tag{2.14}$$

where we introduced the notation $\tilde{g} := g \circ \boldsymbol{p}_h$ for the pullback of g from $\partial\Omega$ to $\partial\Omega_h$.

Using Green's formula we note that the first equation implies that $\lambda_h = -\boldsymbol{n}_h \cdot \nabla u_h$, and therefore we now propose the following modified method: Find $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v \, d\Omega_h + \int_{\partial\Omega_h} \lambda_h v \, ds = \int_{\Omega_h} f^E v \, d\Omega_h \quad \forall v \in V_h \tag{2.15}$$

$$\int_{\partial\Omega_h} u_h \mu \, ds - \int_{\partial\Omega_h} \varrho_h \lambda_h \mu \, ds = \int_{\partial\Omega} \tilde{g} \mu \, ds \quad \forall \mu \in \Lambda_h \tag{2.16}$$

or

$$A(u_h, \lambda_h; v, \mu) = (f^E, v)_{\Omega_h} + (\tilde{g}, \mu)_{\partial\Omega_h} \quad \forall (u_h, \lambda_h) \in V_h \times \Lambda_h \tag{2.17}$$

where $(\cdot, \cdot)_M$ denotes the L_2 scalar product over M, with $\|\cdot\|_M$ the corresponding L_2 norm, and

$$A(u, \lambda; v, \mu) := (\nabla u, \nabla v)_{\Omega_h} + (\lambda, v)_{\partial\Omega_h} + (u, \mu)_{\partial\Omega_h} - (\varrho_h \lambda, \mu)_{\partial\Omega_h}. \tag{2.18}$$

Introducing the triple norm

$$|||(v, \mu)||| := \|\nabla v\|_{\Omega_h} + \left\| h^{\frac{1}{2}} v \right\|_{\partial\Omega_h} + \left\| h^{\frac{1}{2}} \mu \right\|_{\partial\Omega_h}.$$

we have, for all $(w, \varsigma), (v, \mu) \in H^1(\Omega_h) \times L^2(\partial\Omega_h)$, using the Cauchy–Schwarz inequality

$$A(w, \varsigma; v, \mu) \leq \|\nabla w\|_{\Omega_h} \|\nabla v\|_{\Omega_h} + \left\| h^{\frac{1}{2}} \varsigma \right\|_{\partial\Omega_h} \left\| h^{-\frac{1}{2}} v \right\|_{\partial\Omega_h} + \left\| h^{\frac{1}{2}} \mu \right\|_{\partial\Omega_h} \left\| h^{-\frac{1}{2}} w \right\|_{\partial\Omega_h}$$

$$+ \left\| \delta_h h^{-\frac{1}{2}} \varsigma \right\|_{\partial\Omega_h} \|\mu\|_{\partial\Omega_h} \lesssim |||(w, \varsigma)||| \, |||(v, \mu)||| \tag{2.19}$$

where we used that $\delta_h = O(h)$ in the last inequality.

2.5 Relation to Nitsche's method with boundary value correction

Problem (2.17) can equivalently be formulated as finding the stationary points of the Lagrangian

$$\mathcal{L}(u, \lambda) := \frac{1}{2} \|\nabla u\|_{\Omega_h}^2 + (\lambda, u)_{\partial\Omega_h} - \left\| \varrho_h^{1/2} \lambda \right\|_{\partial\Omega_h}^2 - (f^E, u)_{\Omega_h} - (\tilde{g}, \lambda)_{\partial\Omega_h} \tag{2.20}$$

We now follow [12] and add a consistent penalty term and seek stationary points of the augmented Lagrangian

$$\mathcal{L}_{\text{aug}}(u, \lambda) := \mathcal{L}(u, \lambda) + \frac{1}{2} \left\| \gamma^{1/2} (u - \varrho_h \lambda - \tilde{g}) \right\|_{\partial\Omega_h}^2 \tag{2.21}$$

where $\gamma > 0$ remains to be chosen. The corresponding optimality system is

$$(f^E, v)_{\Omega_h} + (\tilde{g}, \mu)_{\partial \Omega_h} = A(u_h, \lambda_h; v, \mu) + (\gamma (u_h - \varrho_h \lambda_h - \tilde{g}), v)_{\partial \Omega_h}$$
$$-(\gamma \varrho_h (u_h - \varrho_h \lambda_h - \tilde{g}), \mu)_{\partial \Omega_h} \qquad (2.22)$$

Now, formally replacing λ_h by $-\boldsymbol{n}_h \cdot \nabla u_h$ and μ by $-\boldsymbol{n}_h \cdot \nabla v$ we obtain

$$(f^E, v)_{\Omega_h} - (\tilde{g}, \boldsymbol{n}_h \cdot \nabla v)_{\partial \Omega_h} = (\nabla u_h, \nabla v)_{\Omega_h} - (\boldsymbol{n}_h \cdot \nabla u_h, v)_{\partial \Omega_h}$$
$$- (u_h, \boldsymbol{n}_h \cdot \nabla v)_{\partial \Omega_h} - (\varrho_h \boldsymbol{n}_h \cdot \nabla u_h, \boldsymbol{n}_h \cdot \nabla v)_{\partial \Omega_h}$$
$$+ (\gamma (u_h + \varrho_h \boldsymbol{n}_h \cdot \nabla u_h - \tilde{g}), v + \varrho_h \boldsymbol{n}_h \cdot \nabla v)_{\partial \Omega_h}$$
$$(2.23)$$

Setting now $\gamma = \gamma_0/h$, with γ_0 sufficiently large to ensure coercivity, we obtain the symmetrized version of the boundary value corrected Nitsche method proposed in [1] with optimal convergence up to order $p = 3$. Observe that the form remains positive for large positive ρ_h, but the control of the trace of u_h degenerates for large ρ_h, so stability depends on ρ_h, but there is no strict bound for large values of ρ_h. This means that $\partial \Omega_h$ has to be a reasonably good approximation of $\partial \Omega$, or Ω is approximated from the inside. We define this method as: Find $\tilde{u}_h \in V_h$ such that

$$A_{Nit}(\tilde{u}_h, v_h) = (f^E, v_h)_{\partial \Omega_h} + (\tilde{g}, \boldsymbol{n}_h \cdot \nabla v_h)_{\partial \Omega_h} + (\gamma \tilde{g}, v_h + \varrho_h \boldsymbol{n}_h \cdot \nabla v_h)_{\partial \Omega_h} \quad (2.24)$$

for all $v_h \in V_h$. Here the bilinear form is defined by

$$A_{Nit}(w_h, v_h) := (\nabla w_h, \nabla v_h)_{\Omega_h} - (\boldsymbol{n}_h \cdot \nabla w_h, v_h + \varrho_h \boldsymbol{n}_h \cdot \nabla v_h)_{\partial \Omega_h}$$
$$- (w_h + \varrho_h \boldsymbol{n}_h \cdot \nabla w_h, \boldsymbol{n}_h \cdot \nabla v)_{\partial \Omega_h} + (\varrho_h \boldsymbol{n}_h \cdot \nabla w_h, \boldsymbol{n}_h \cdot \nabla v)_{\partial \Omega_h}$$
$$+ (\gamma (w_h + \varrho_h \boldsymbol{n}_h \cdot \nabla w_h, v + \varrho_h \boldsymbol{n}_h \cdot \nabla v_h)_{\partial \Omega_h}. \qquad (2.25)$$

In [1,3] the following error estimate was proved:

Theorem 2.1 *Let u be the solution to* (2.1)–(2.2) *and* $u_h \in V_h$ *the solution to* (2.24), *with* γ *sufficiently large, then for sufficiently smooth u there holds*

$$\|\nabla (u^E - \tilde{u}_h)\|_{\Omega_h} + \left\| h^{\frac{1}{2}} \boldsymbol{n}_h \cdot \nabla (u^E - \tilde{u}_h) \right\|_{\partial \Omega_h} + \left\| h^{-\frac{1}{2}} (\tilde{u}_h + \varrho \boldsymbol{n}_h \cdot \nabla \tilde{u}_h - \tilde{g}) \right\|_{\partial \Omega_h}$$
$$\lesssim h^k \|u\|_{H^{k+1}(\Omega)} + h^{-1/2} \delta_h^2 \sup_{|t| \leq \delta_0} \|D_{\boldsymbol{n}_h}^2 u^E\|_{L^2(\partial \Omega_t)}$$
$$+ h^{1/2} \delta_h^{l+1} \sup_{0 \leq t \leq \delta_0} \|D_{\boldsymbol{n}_h}^l (f^E + \Delta u^E)\|_{L^2(\partial \Omega_t)}, \qquad (2.26)$$

here $\partial \Omega_t = \{\boldsymbol{x} \in U_{\delta_0}(\partial \Omega) : \varrho(\boldsymbol{x}) = t\}$ *and l is an integer larger than or equal to zero. The hidden constant depends on the parameter* γ.

As we shall see below, the multiplier method satisfies the same estimate, but is independent of any parameter.

3 Elements of analysis

In this section we will prove some basic results on the stability and the accuracy of the method (2.17). We define the space of piecewise polynomial functions of order less than or equal to l on the trace mesh $\partial\Omega_h$ by

$$X_h^l := \{\mu \in L^2(\partial\Omega_h) : \mu \in \mathbb{P}_l(F), \ \forall F \in \partial\Omega_h\}.$$

We will use X_h^l to define the multiplier space. For the bulk variable we let V_h be the space of continuous piecewise polynomial functions of order k, enriched with higher order bubbles on the faces in $\partial\Omega_h$ so that inf-sup stability holds when combined with the multiplier space $\Lambda_h := X_h^{k-1}$. More precisely we assume that there exists constants c_0 and c_1 such that for every $\eta_h \in \Lambda_h$ there exists $v_\eta \in V_h$ such that

$$c_0 \left\| h^{\frac{1}{2}} \eta_h \right\|_{\partial\Omega_h}^2 \leq (\eta_h, v_\eta)_{\partial\Omega_h} \quad \text{and} \quad \|\nabla v_\eta\|_{\Omega_h} + \left\| h^{-\frac{1}{2}} v_\eta \right\|_{\partial\Omega_h} \leq c_1 \left\| h^{\frac{1}{2}} \eta_h \right\|_{\partial\Omega_h}. \tag{3.1}$$

For details on stable choices of the spaces we refer to [13–15].

For the analysis we will make regular use of the following standard trace and inverse inequalities, for all elements $K \in \mathcal{K}_h$

$$\|v\|_{\partial K} \lesssim h_K^{-\frac{1}{2}} \|v\|_K + h_K^{\frac{1}{2}} \|\nabla v\|_K, \quad \text{for all } v \in H^1(K), \tag{3.2}$$

$$h_K^{\frac{1}{2}} \|v_h\|_{\partial K} + h_K \|\nabla v_h\|_K \lesssim \|v_h\|_K, \quad \text{for all } v_h \in \mathbb{P}_k(K). \tag{3.3}$$

We also recall the following bound from [3],

$$\|v_h\|_{\Omega_h \setminus \Omega} \lesssim \delta_h^{\frac{1}{2}} \left(h^{\frac{1}{2}} + \delta_h^{\frac{1}{2}} \right) \left(\|\nabla v_h\|_{\Omega_h} + \left\| h^{-\frac{1}{2}} v_h \right\|_{\partial\Omega_h} \right). \tag{3.4}$$

We let $\pi_h : L^2(\partial\Omega_h) \to \Lambda_h$ denote the L^2-orthogonal projection, which has optimal approximation in the L^2-norm, and $i_h : H^2(\Omega_h) \to V_h$ the standard Lagrange interpolant. We can then prove the following approximation property

Lemma 3.1 *For all $v \in H^{k+1}(\Omega)$ there holds, with $i_h v^E \in V_h$ and $\pi_h \lambda^E \in \Lambda_h$,*

$$\||(v^E - i_h v^E, \lambda^E - \pi_h \lambda^E)|\| \lesssim h^k \|u\|_{H^{k+1}(\Omega)}.$$

where $\lambda^E := \boldsymbol{n}_h \cdot \nabla u^E|_{\partial\Omega_h}$.

Proof It is straightforward to show using standard interpolation that the following approximation result is satisfied,

$$\||(v^E - i_h v^E, \lambda^E - \pi_h \lambda^E)|\| \lesssim h^k \|u\|_{H^{k+1}(\Omega_h)}.$$

First note that by using the trace inequality (3.2) locally on each F on $\partial\Omega_h$

$$\left\| h^{-\frac{1}{2}}(v^E - i_h v^E) \right\|_{\partial\Omega_h} \lesssim h^{-1} \| v^E - i_h v^E \|_{\Omega_h} + \| \nabla(v^E - i_h v^E) \|_{\Omega_h}.$$

It follows, using standard interpolation and the stability of the extension, that

$$\| \nabla(v^E - i_h v^E) \|_{\Omega_h} + \left\| h^{-\frac{1}{2}}(v^E - i_h v^E) \right\|_{\partial\Omega_h} \lesssim h^k \| u \|_{H^{k+1}(\Omega)}.$$

Then note that by using standard interpolation locally on each face F on $\partial\Omega_h$, (see e.g. [16, Lemma 5.2]),

$$\| \lambda^E - \pi_h \lambda^E \|_F \lesssim h^{k-\frac{1}{2}} | \boldsymbol{n}_h \cdot \nabla u^E |_{H^{k-\frac{1}{2}}(F)} \lesssim h^{k-\frac{1}{2}} | u^E |_{H^{k+\frac{1}{2}}(F)},$$

we have

$$\| \lambda^E - \pi_h \lambda^E \|_{\partial\Omega_h} \lesssim h^{k-\frac{1}{2}} | u |_{H^{k+\frac{1}{2}}(\partial\Omega_h)}.$$

Using a global trace inequality, $| u^E |_{H^{k+\frac{1}{2}}(\partial\Omega_h)} \lesssim \| u^E \|_{H^{k+1}(\partial\Omega_h)}$, we see that

$$\left\| h^{\frac{1}{2}}(\lambda^E - \pi_h \lambda^E) \right\|_{\partial\Omega_h} \lesssim h^k \| u^E \|_{H^{k+1}(\Omega_h)}.$$

We conclude by applying the stability estimate for the extension (2.11). $\square$

We will now state an elementary lemma showing that the approximation of the bulk variable using the trace variable is optimal in $H^1(\Omega_h)$.

Lemma 3.2 *Let $v_h \in V_h$ and let $\mu_v = \pi_h(v_h|_{\partial\Omega}) \in \Lambda_h$. Then there holds:*

$$\left\| h^{-\frac{1}{2}}(v_h - \mu_v) \right\|_{\partial\Omega_h} \leq C_t \| \nabla v_h \|_{\Omega_h}. \tag{3.5}$$

Proof We recall the bound

$$\| v - \pi_h v \|_{\partial\Omega_h} \lesssim h \| \nabla_\partial v \|_{\partial\Omega_h} \tag{3.6}$$

for all $v \in H^1(\partial\Omega_h)$ and where ∇_∂ denotes the gradient along the boundary. Using (3.6) we see that

$$\left\| h^{-\frac{1}{2}}(v_h - \mu_v) \right\|_{\partial\Omega_h} \lesssim h^{\frac{1}{2}} \| \nabla_\partial v_h \|_{\partial\Omega_h}.$$

The result follows by applying the trace inequality, similar to (3.3), $\| \nabla_\partial v_h \|_{\partial K} \lesssim H^{-\frac{1}{2}} \| \nabla v_h \|_K$. $\square$

The formulation (2.17) satisfies the following stability result

Proposition 3.1 *Assume that for two constants $C_{\partial\Omega^-}$, $C_{\partial\Omega^+}$,*

$$0 < C_{\partial\Omega^-} \leq \min\left(c_0^2/(8c_1^2),\, C_t c_0/(2c_1)\right) \quad \text{and} \quad C_{\partial\Omega^+} > 0$$

there holds $-C_{\partial\Omega^-} h_{min} \leq \varrho_h \leq C_{\partial\Omega^+} h$ and that $V_h \times \Lambda_h$ satisfies the stability condition (3.1). Then for every $(y_h, \eta_h) \in V_h \times \Lambda_h$ there exists $(v_h, \mu_h) \in V_h \times \Lambda_h$ such that

$$\|| (y_h, \eta_h) \||^2 \lesssim A(y_h, \eta_h; v_h, \mu_h) \tag{3.7}$$

and

$$\|| (v_h, \mu_h) \|| \lesssim \|| (y_h, \eta_h) \|| . \tag{3.8}$$

Proof First observe that

$$A(y_h, \eta_h; y_h, -\eta_h) = \|\nabla y_h\|^2_{\Omega_h} + (\varrho_h \eta_h, \eta_h)_{\partial\Omega_h}. \tag{3.9}$$

Then recall that since the space satisfies the inf-sup condition for $\eta_h \in \Lambda_h$ there exists $v_\eta \in V_h$ so that (3.1) holds. Then note that

$$A(y_h, \eta_h; c_\eta v_\eta, 0) = (\nabla y_h, \nabla c_\eta v_\eta)_{\Omega_h} + (\eta_h, c_\eta v_\eta)_{\partial\Omega_h} = I + II.$$

Using the definition of v_η and the bounds of (3.1) we have the following bounds for the terms I and II of the right hand side

$$I \geq -\|\nabla y_h\|_{\Omega_h} c_\eta \|\nabla v_\eta\|_{\Omega_h} \geq -\frac{1}{4}\|\nabla y_h\|^2_{\Omega_h} - c_\eta^2 c_1^2 \left\| h^{\frac{1}{2}} \eta_h \right\|^2_{\partial\Omega}.$$

$$II = (\eta_h, c_\eta v_\eta)_{\partial\Omega_h} \geq c_0 c_\eta \left\| h^{\frac{1}{2}} \eta_h \right\|^2_{\partial\Omega_h}.$$

Therefore

$$A(y_h, \eta_h; y_h + c_\eta v_\eta, -\eta_h) \geq \frac{3}{4}\|\nabla y_h\|^2_{\Omega_h} + c_\eta(c_0 - c_\eta c_1^2) \left\| h^{\frac{1}{2}} \eta_h \right\|^2_{\partial\Omega}$$
$$+ ([\varrho_h]_+ \eta_h, \eta_h)_{\partial\Omega_h} - ([\varrho_h]_- \eta_h, \eta_h)_{\partial\Omega_h},$$

where $[\varrho]_\pm = \frac{1}{2}(|\varrho| \pm \varrho)$. It follows that for $c_\eta = c_0/(2c_1^2)$, $C_{\partial\Omega^-} \leq c_0^2/(8c_1^2)$,

$$\frac{3}{4}\|\nabla y_h\|^2_{\Omega_h} + c_0^2/(8c_1^2) \left\| h^{\frac{1}{2}} \eta_h \right\|^2_{\partial\Omega}$$
$$\leq \frac{3}{4}\|\nabla y_h\|^2_{\Omega_h} + (c_\eta(c_0 - c_\eta c_1^2) - C_{\partial\Omega^-}) \left\| h^{\frac{1}{2}} \eta_h \right\|^2_{\partial\Omega}$$
$$+ ([\varrho_h]_+ \eta_h, \eta_h)_{\partial\Omega_h} \leq A(y_h, \eta_h; y_h + c_\eta v_\eta, -\eta_h). \tag{3.10}$$

Observe that $c_\eta(c_0 - c_\eta c_1^2) - C_{\partial\Omega^-} \geq c_0^2/(8c_1^2)$. Finally let $\mu_y = \pi_h y_h$ and observe that since $\varrho_h \lesssim h$,

$$
\begin{aligned}
A(y_h, \eta_h; 0, \mu_y) &= (y_h, h^{-1}\mu_y)_{\partial\Omega_h} - (\varrho_h \eta_h, h^{-1}\mu_y)_{\partial\Omega_h} \\
&\geq \left\| h^{-\frac{1}{2}} y_h \right\|_{\partial\Omega_h}^2 - \left\| h^{-\frac{1}{2}}(y_h - \mu_y) \right\|_{\partial\Omega_h}^2 - \frac{1}{\varepsilon} \left\| [\varrho_h]_+^{\frac{1}{2}} \eta_h \right\|_{\partial\Omega_h}^2 \\
&\quad - C_{\partial\Omega^-}^2 \left\| h^{\frac{1}{2}} \eta_h \right\|_{\partial\Omega_h}^2 - \left(\frac{1}{4} + \frac{\varepsilon}{4} \| [\varrho_h]_+ \|_{L^\infty(\partial\Omega_h)} h^{-1} \right) \left\| h^{-\frac{1}{2}} \mu_y \right\|_{\partial\Omega_h}^2 \\
&\geq \frac{1}{2} \left\| h^{-\frac{1}{2}} y_h \right\|_{\partial\Omega_h}^2 - C_t^2 \| \nabla y_h \|_{\Omega_h}^2 \\
&\quad - C_{\partial\Omega^-}^2 \left\| h^{\frac{1}{2}} \eta_h \right\|_{\partial\Omega_h}^2 - C_{\partial\Omega^+} \left\| [\varrho_h]_+^{\frac{1}{2}} \eta_h \right\|_{\partial\Omega_h}^2 .
\end{aligned}
\tag{3.11}
$$

Where the last inequality follows by applying Lemma 3.2, fixing $\varepsilon = h/\| [\varrho_h]_+ \|_{L^\infty(\partial\Omega_h)}$.

The bound (3.7) then follows by taking $v_h = y_h + c_\eta v_\eta$ and $\mu_h = -\eta_h + c_y h^{-1}\mu_y$ with $c_\eta = c_0/(2c_1^2)$ and $c_y = \min((4C_t^2)^{-1}, C_{\partial\Omega^+}^{-1})$ and assuming that $C_{\partial\Omega^-} < C_t c_0/(2c_1)$. This results in the bound

$$
\begin{aligned}
A(y_h, \eta_h; y_h + c_\eta v_\eta, -\eta_h + c_y \mu_y) &\geq \underbrace{\left(\frac{3}{4} - c_y C_t^2 \right)}_{\geq \frac{1}{2}} \| \nabla y_h \|_{\Omega_h}^2 + \frac{c_y}{2} \left\| h^{-\frac{1}{2}} y_h \right\|_{\partial\Omega_h}^2 \\
&\quad + \underbrace{\left(c_0^2/(8c_1^2) - c_y C_{\partial\Omega^-}^2 \right)}_{\geq (c_0^2/(16c_1^2))} \left\| h^{\frac{1}{2}} \eta_h \right\|_{\partial\Omega_h}^2 + \underbrace{(1 - c_y C_{\partial\Omega^+})}_{\geq 0} ([\varrho_h]_+ \eta_h, \eta_h)_{\partial\Omega_h} \\
&\geq \min\left(1/2, c_0^2/(16c_1^2), c_y/2 \right) ||| (y_h, \eta_h) |||^2 .
\end{aligned}
$$

To conclude the proof we need to show that

$$
||| (v_h, \mu_h) ||| \lesssim ||| (y_h, \eta_h) ||| .
\tag{3.12}
$$

By the triangle inequality we have

$$
||| (v_h, \mu_h) ||| \leq ||| (y_h, \eta_h) ||| + ||| (c_\eta v_\eta, c_y h^{-1}\mu_y) ||| .
\tag{3.13}
$$

By definition

$$
||| (c_\eta v_\eta, c_y h^{-1}\mu_y) ||| = c_\eta \| \nabla v_\eta \|_{\Omega_h} + c_\eta \left\| h^{-\frac{1}{2}} v_\eta \right\|_{\partial\Omega_h} + c_y \left\| h^{-\frac{1}{2}} \mu_y \right\|_{\partial\Omega_h}
\tag{3.14}
$$

and the proof follows from (3.1) together with the stability of π_h in L^2. $\qquad\square$

Remark 3.1 It follows from Proposition 3.1 that $C_{\partial\Omega^-}$ must respect a bound depending on the stability constants of the finite element pair as defined in (3.1) and the constant

of the approximation bound in Lemma 3.2. $C_{\partial\Omega+}$ however is free, but as it grows the control of $\|h^{-\frac{1}{2}} y_h\|_{\partial\Omega_h}$ degenerates, since c_y has to be taken smaller than $C_{\partial\Omega+}^{-1}$.

An immediate consequence of the Proposition 3.1 is the existence of a unique discrete solution to the formulation (2.17). We will now use this stability result to prove an error estimate. First we prove some preliminary lemmas quantifying the consistency error induced by using the approximate domain Ω_h and the first order Taylor expansion.

Lemma 3.3 *Let u be the solution to (2.1)–(2.2), and $(u_h, \lambda_h) \in V_h \times \Lambda_h$ the solution to (2.17) then there holds for all $(v_h, \mu_h) \in V_h \times \Lambda_h$,*

$$A(u^E - u_h, \lambda^E - \lambda_h; v_h, \mu_h) = -(f^E + \Delta u^E, v_h)_{\Omega_h \setminus \Omega} - (T_2^\Delta(u), \mu_h)_{\partial\Omega_h},$$

where u^E, f^E denotes the extension of u, f.

Proof First observe that by the definition (2.17)

$$A(u^E - u_h, \lambda^E - \lambda_h; v_h, \mu_h) = A(u^E, \lambda^E; v_h, \lambda_h) - (f^E, v_h)_{\Omega_h} - (\tilde{g}, \mu_h)_{\partial\Omega_h}.$$

Integrating by parts we then obtain

$$A(u^E, \lambda^E; v_h, 0) = (-\Delta u^E, v_h)_{\Omega_h} + (\nabla u^E + \lambda^E, v_h)_{\partial\Omega_h} = (-\Delta u^E, v_h)_{\Omega_h}.$$

Using that $f + \Delta u = 0$ in Ω we have that

$$-(\Delta u^E + f^E, v_h)_{\Omega_h} = -(\Delta u^E + f^E, v_h)_{\Omega_h \setminus \Omega}.$$

Considering now the boundary term we have

$$A(u^E, \lambda^E; 0, \mu_h) = (u^E - \varrho_h \lambda^E - (\tilde{g}, \mu_h)_{\partial\Omega_h} = -(T_2^\Delta(u), \mu_h)_{\partial\Omega_h}$$

where we used that

$$\tilde{g}|_{\partial\Omega_h} = u \circ p_h|_{\partial\Omega_h} = u^E|_{\partial\Omega_h} + \varrho_h n_h \cdot \nabla u^E|_{\partial\Omega_h} + \int_0^{\varrho_h} \int_0^s \partial_t^2 u^E(x + t n_h) \, dt ds.$$

$\square$

Lemma 3.4 *Let u be a sufficiently smooth solution to (2.1)–(2.2), then the following bounds are satisfied*

$$\|Ef + \Delta u^E\|_{\Omega_h \setminus \Omega} \lesssim \delta_h^{l+\frac{1}{2}} \sup_{0 \leq t \leq \delta_0} \|D_{n_h}^l (f^E + \Delta u^E)\|_{L^2(\partial\Omega_t)}, \quad l \geq 0, \quad (3.15)$$

$$\|T_2^\Delta(u)\|_{\partial\Omega_h} \lesssim \delta_h^2 \sup_{|t| \leq \delta_0} \|D_{n_h}^2 u^E\|_{L^2(\partial\Omega_t)}. \quad (3.16)$$

Proof For the proof of (3.15) we refer to [3]. The proof of (3.16) follows using Cauchy–Schwarz inequality repeatedly

$$\|T_2^{\Delta}(u)\|_{\partial\Omega_h}^2 = \int_{\partial\Omega_h} \left(\int_0^{\varrho_h} \int_0^s \partial_t^2 u^E(\boldsymbol{x} + t\boldsymbol{n}_h) \, dt ds \right)^2 dx,$$

$$\|T_2^{\Delta}(u)\|_{\partial\Omega_h}^2 \le \int_{\partial\Omega_h} |\varrho_h|^3 \|D_{\boldsymbol{n}_h}^2 u^E(\boldsymbol{x} + s\boldsymbol{n}_h)\|_{[-|\varrho_h|,|\varrho_h|]}^2 \, dx \le \delta_h^3 \|D_{\boldsymbol{n}_h}^2 u^E\|_{U_{\delta_h}}^2,$$

and finally we conclude observing that

$$\|D_{\boldsymbol{n}_h}^2 u^E\|_{U_{\delta_h}} \le \delta_h^{\frac{1}{2}} \sup_{|t| \le \delta_0} \|D_{\boldsymbol{n}_h}^2 u^E\|_{L^2(\partial\Omega_t)}.$$

□

Theorem 3.1 *Let* $u \in H^{k+1}(\Omega)$ *denote the solution to* (2.4)–(2.5). *Let* $u_h, \lambda_h \in V_h \times \Lambda_h$ *denote the solution of* (2.17). *Assume that the polynomial order of* V_h *is* $k \in \{1, 2, 3\}$, *with enrichment on the boundary and* $\Lambda_h \equiv X_h^{k-1}$. *Assume that the hypothesis of Proposition* 3.1 *are satisfied. Then there holds,*

$$\||(u^E - u_h, \lambda^E - \lambda_h)\|| \lesssim h^k \|u\|_{H^{k+1}(\Omega)} + h^{\frac{1}{2}} \delta_h^{l+1} \sup_{0 \le t \le \delta_0} \|D_{\boldsymbol{n}_h}^l (f + \Delta u^E)\|_{L^2(\partial\Omega_t)}$$

$$+ h^{-1/2} \delta_h^2 \sup_{|t| \le \delta_0} \|D_{\boldsymbol{n}_h}^2 u^E\|_{L^2(\partial\Omega_t)} \tag{3.17}$$

Proof We consider the discrete errors $e_h = u_h - v_h$ and $\varsigma_h = \lambda_h - \zeta_h$ for some $v_h, \zeta_h \in V_h \times \Lambda_h$. Using the triangle inequality and we have

$$\||(u^E - u_h, \lambda^E - \lambda_h)\|| \le \||(u^E - v_h, \lambda^E - \zeta_h)\|| + \||(e_h, \varsigma_h)\||. \tag{3.18}$$

By choosing v_h judiciously the first term on the right hand side is bounded by standard interpolation. We therefore only need to consider the second term. By the stability estimate of Proposition 3.1 we have

$$\||(e_h, \varsigma_h)\||^2 \lesssim A(e_h, \varsigma_h; v_h, \mu_h). \tag{3.19}$$

Using Lemma 3.3 we find that

$$A(e_h, \varsigma_h; v_h, \mu_h) = A(u - v_h, \lambda - \zeta_h; v_h, \mu_h)$$
$$+ (f^E + \Delta Eu, v_h)_{\Omega_h \setminus \Omega} + (T_2^{\Delta}(u), \mu_h)_{\partial\Omega_h}$$

Applying the continuity of the form A, (2.19) in the first term of the right hand side, the Cauchy–Schwarz inequality followed by the inequality (3.4) in the second and finally an h-weighted Cauchy–Schwarz inequality in the third we obtain the bound

$$A(e_h, \varsigma_h; v_h, \mu_h)$$

$$\lesssim \||(u - v_h, \lambda - \zeta_h)\|| \, \||(v_h, \mu_h)\|| + h^{\frac{1}{2}} \delta_h^{\frac{1}{2}} \|f^E + \Delta u^E\|_{\Omega_h \setminus \Omega} \, \||(v_h, 0)\||$$

$$+ \left\| h^{-\frac{1}{2}} T_2^{\Delta}(u) \right\|_{\partial \Omega_h} \||(0, \mu_h)\||$$

$$\lesssim \left(\||(u - v_h, \lambda - \zeta_h)\|| + h^{\frac{1}{2}} \delta_h^{\frac{1}{2}} \|f^E + \Delta u^E\|_{\Omega_h \setminus \Omega} + \left\| h^{-\frac{1}{2}} T_2^{\Delta}(u) \right\|_{\partial \Omega_h} \right) \||(v_h, \mu_h)\||.$$

It then follows from equation (3.18) and (3.1) that

$$\||(u^E - u_h, \lambda^E - \lambda_h)\|| \lesssim \inf_{(v_h, \mu_h) \in V_h \times \Lambda_h} \||(u^E - v_h, \lambda^E - \mu_h)\||$$

$$+ h^{\frac{1}{2}} \delta_h^{\frac{1}{2}} \|f^E + \Delta u^E\|_{\Omega_h \setminus \Omega} + \left\| h^{-\frac{1}{2}} T_2^{\Delta}(u) \right\|_{\partial \Omega_h}.$$

To conclude we choose $v_h = i_h u^E$ and $\mu_h = \pi_h \lambda^E$ and apply the bounds of Lemmas 3.1 and 3.4 to obtain

$$\||(u^E - u_h, \lambda^E - \lambda_h)\|| \lesssim h^k |u|_{H^{k+1}(\Omega)} + h^{\frac{1}{2}} \delta_h^{l+1} \sup_{0 \leq t \leq \delta_0} \|D_{\boldsymbol{n}_h}^l (f^E + \Delta u^E)\|_{L^2(\partial \Omega_t)}$$

$$+ h^{-\frac{1}{2}} \delta_h^2 \sup_{|t| \leq \delta_0} \|D_{\boldsymbol{n}_h}^2 u^E\|_{L^2(\partial \Omega_t)}.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3.1 *Let $l = 0$ for $k = 1$ and $l = 1$ for $k = 2, 3$. Assume that, $\delta_h \lesssim O(h^{(2k+1)/4})$; then under the same assumptions as Proposition 3.1 and Theorem 3.1 there holds:*

$$\||(u^E - u_h, \lambda^E - \lambda_h)\|| \lesssim h^k \left(\|u\|_{H^{k+1}(\Omega)} + \|u\|_{H^{l+\frac{5}{2}+\epsilon}(\Omega)} + \|u\|_{H^{\frac{5}{2}}(\Omega)} \right), \quad k = 1, 2, 3,$$

(3.20)

where $\epsilon > 0$ when $l = 0$ and $\epsilon = 0$ when $l = 1$.

Proof First consider the case $l = 0$. Then using a trace inequality followed by the triangle inequality

$$\|(f^E + \Delta u^E)\|_{L^2(\partial \Omega_t)} \lesssim \|f^E + \Delta u^E\|_{H^{\frac{1}{2}+\epsilon}(U_{\delta_0}(\Omega))}$$

$$\lesssim \|f^E\|_{H^{\frac{1}{2}+\epsilon}(U_{\delta_0}(\Omega))} + \|\Delta u^E\|_{H^{\frac{1}{2}+\epsilon}(U_{\delta_0}(\Omega))}.$$

Using the stability of the extension

$$\|f^E\|_{H^{\frac{1}{2}+\epsilon}(U_{\delta_0}(\Omega))} + \|\Delta u^E\|_{H^{\frac{1}{2}+\epsilon}(U_{\delta_0}(\Omega))} \lesssim \|f\|_{H^{\frac{1}{2}+\epsilon}(\Omega)} + \|u\|_{H^{\frac{5}{2}+\epsilon}(\Omega)}$$

$$\lesssim \|u\|_{H^{\frac{5}{2}+\epsilon}(\Omega)}.$$

Similarly

$$\|D^2_{\boldsymbol{n}_h} u^E\|_{L^2(\partial\Omega_t)} \lesssim \|u\|_{H^{\frac{5}{2}}(\Omega)}.$$

For $l = 1$ the following bound holds by the same argument

$$\|D_{\boldsymbol{n}_h}(f^E + \Delta u^E)\|_{L^2(\partial\Omega_t)} \lesssim \|u\|_{H^{\frac{7}{2}}(\Omega)}.$$

Consider now the powers of h

$$h^{\frac{1}{2}}\delta_h^{l+1} \lesssim \begin{cases} h^{\frac{5}{4}} & \text{when } k = 1, l = 0 \\ h^3 & \text{when } k = 2, l = 1 \\ h^4 & \text{when } k = 3, l = 1. \end{cases}$$

and

$$h^{-\frac{1}{2}}\delta_h^2 \lesssim \begin{cases} h & \text{when } k = 1 \\ h^2 & \text{when } k = 2 \\ h^3 & \text{when } k = 3. \end{cases}$$

$\square$

Remark 3.2 We see that optimal convergence is obtained when $\delta_h \lesssim h^{(2k+1)/4}$. It follows that, for the case $k = 1$, $\delta_h = O(h^{3/4})$ is sufficient for optimal accuracy. Such a poor geometry approximation however violates the condition $\delta_h = O(h)$ necessary for the stability of Proposition 3.1 to hold uniformly.

We now prove an error estimate in the L^2-norm

Theorem 3.2 *Under the same assumptions as for Theorem* 3.1, *there holds for* $l \geq 0$,

$$\|u - u_h\|_{\Omega\cap\Omega_h} \lesssim h^{k+1}|u|_{H^{k+1}(\Omega)} + h^{\frac{1}{2}}\delta_h^{l+1} \sup_{0 \leq t \leq \delta_0} \|D^l_{\boldsymbol{n}_h}(f + \Delta u^E)\|_{L^2(\partial\Omega_t)}$$

$$+\delta_h^2 \sup_{|t| \leq \delta_0} \|D^2_{\boldsymbol{n}_h} u^E\|_{L^2(\partial\Omega_t)}$$

Proof Let $\varphi \in H_0^1(\Omega)$ be the solution to the dual problem

$$-\Delta\varphi = \psi \text{ in } \Omega$$

where $\psi|_{\Omega_h} = u^E - u_h$ and $\psi|_{\Omega\setminus\Omega_h} = 0$. By (2.3) there holds $\|\varphi\|_{H^2(\Omega)} \lesssim \|u - u_h\|_{\Omega_h\cap\Omega}$. Now observe that, if $e = u^E - u_h$ and $\eta = \lambda^E - \lambda_h$ then

$$\|e\|^2_{\Omega_h} = (e, \psi + \Delta\varphi^E)_{\Omega_h} - (e, \Delta\varphi^E)_{\Omega_h}$$
$$= (e, \psi + \Delta\varphi^E)_{\Omega_h\setminus\Omega} + (\nabla e, \nabla\varphi^E)_{\Omega_h} - (e, \boldsymbol{n}_h \cdot \nabla\varphi^E)_{\partial\Omega_h} = I + II + III.$$

For the first term on the right hand side we apply the Cauchy–Schwarz inequality and the boundary Poincaré inequality [3]

$$\|v\|_{\Omega_h \setminus \Omega} \lesssim \delta_h \|\mathbf{n}_h \cdot \nabla v\|_{\Omega_h \setminus \Omega} + \delta_h^{\frac{1}{2}} \|v\|_{\partial\Omega_h}^2$$

$$\leq \delta_h^{\frac{1}{2}} \left(\delta_h^{\frac{1}{2}} + h^{\frac{1}{2}} \right) \||(v,0)|\|, \quad v \in H^1(\Omega_h \setminus \Omega)$$

to obtain

$$I = (e, \psi + \Delta\varphi^E)_{\Omega_h \setminus \Omega}$$

$$\lesssim \|e\|_{\Omega_h \setminus \Omega} \left(\|\psi\|_{\Omega_h \setminus \Omega} + \|\Delta\varphi^E\|_{\Omega_h \setminus \Omega} \right) \lesssim \|e\|_{\Omega_h \setminus \Omega} \left(\|e\|_{\Omega_h \setminus \Omega} + \|\varphi\|_{H^2(\Omega)} \right)$$

$$\lesssim \delta_h^{\frac{1}{2}} \left(\delta_h^{\frac{1}{2}} + h^{\frac{1}{2}} \right) \||(e,0)|\| \|e\|_{\Omega_h} \lesssim \delta_h^{\frac{1}{2}} h^{\frac{1}{2}} \||(e,0)|\| \|e\|_{\Omega_h}.$$

To bound the terms *II* and *III* we apply the consistency of Lemma 3.3 with $v_h = i_h\varphi^E$ and $\mu_h = \pi_h \mathbf{n}_h \nabla\varphi^E$, introducing the notation $\zeta^E := -\mathbf{n}_h \cdot \nabla\varphi^E$

$$II + III = A(e, 0; \varphi^E, \zeta^E) - A(e, \eta; i_h\varphi, \pi_h\zeta)$$
$$- (f^E + \Delta u^E, i_h\varphi)_{\Omega_h \setminus \Omega} - (T_2^\Delta(u), \pi_h\zeta)_{\partial\Omega_h}.$$

For the first two terms on the right hand side we see that, using Lemma 3.3

$$A(e, 0; \varphi^E, \zeta^E) - A(e, \eta; i_h\varphi, \pi_h\zeta^E)$$
$$= A(e, \eta; \varphi^E - i_h\varphi, \zeta^E - \pi_h\zeta^E) - (\eta, \varphi - \rho_h\zeta^E)_{\partial\Omega_h}$$
$$\lesssim \||(e, \eta)|\| \||(\varphi^E - i_h\varphi, \zeta^E - \pi_h\zeta^E)|\|$$
$$+ \left\| h^{\frac{1}{2}}\eta \right\|_{\partial\Omega_h} h^{-\frac{1}{2}} \|\varphi^E + \rho_h\zeta^E\|_{\partial\Omega_h}$$
$$\lesssim h \||(e, \eta)|\| \left(\|\varphi\|_{H^2(\Omega)} + h^{-\frac{3}{2}} \|T_2^\Delta(\varphi)\|_{\partial\Omega_h} \right).$$

Observing that using the arguments to bound the Taylor remainder term,

$$h^{-\frac{3}{2}} \|T_2^\Delta(\varphi)\|_{\partial\Omega_h} \lesssim h^{-\frac{3}{2}} \delta_h^{\frac{3}{2}} \|D_{\mathbf{n}_h}^2 \varphi^E\|_{U_{\delta_h}(\partial\Omega_h)}^2 \lesssim h^{-\frac{3}{2}} \delta_h^{\frac{3}{2}} \|\varphi\|_{H^2(\Omega)}$$

we conclude that under the assumption $\delta_h \lesssim h$ there holds

$$A(e, 0; \varphi^E, \zeta^E) - A(e, \eta; i_h\varphi, \pi_h\zeta^E) \lesssim h \||(e, \eta)|\| \|\varphi\|_{H^2(\Omega)} \lesssim h \||(e, \eta)|\| \|e\|_{\Omega_h}.$$

To conclude the proof we bound the two non-conformity errors. First the bulk term resulting from the geometry mismatch,

$$
\begin{aligned}
(f^E + \Delta u^E, i_h \varphi^E)_{\Omega_h \setminus \Omega} &\lesssim \| f^E + \Delta u^E \|_{\Omega_h \setminus \Omega} \| i_h \varphi^E \|_{\Omega_h \setminus \Omega} \\
&\lesssim \delta_h^{l+\frac{1}{2}} \| D_{\boldsymbol{n}_h}^l (f^E + \Delta u^E) \|_{L^2(\partial \Omega_t)} \delta_h^{\frac{1}{2}} \left(\delta_h^{\frac{1}{2}} + h^{\frac{1}{2}} \right) \ ||| \, (i_h \varphi^E, 0) \, ||| \\
&\lesssim h^{\frac{1}{2}} \delta_h^{l+1} \| D_{\boldsymbol{n}_h}^l (f^E + \Delta u^E) \|_{L^2(\partial \Omega_t)} \| \varphi \|_{H^2(\Omega)},
\end{aligned}
$$

where we used $\| h^{-\frac{1}{2}} i_h \varphi^E \|_{\partial \Omega_h} \lesssim \| h^{-\frac{1}{2}} (i_h \varphi^E - \varphi) \|_{\partial \Omega} + h^{-\frac{1}{2}} \delta_h \| \boldsymbol{n}_h \cdot \nabla i_h \varphi^E \|_{\partial \Omega} \lesssim \| \varphi \|_{H^2(\Omega)}$.

The Taylor term is bounded using equation (3.16),

$$
\begin{aligned}
(T_2^{\Delta}(u), \pi_h \zeta^E)_{\partial \Omega_h} &\lesssim \| T_2^{\Delta}(u) \|_{\partial \Omega_h} \| \pi_h \zeta^E \|_{\partial \Omega_h} \\
&\lesssim \delta_h^2 \| D_{\boldsymbol{n}_h}^2 u^E \|_{L^2(\partial \Omega_t)} \| \zeta^E \|_{\partial \Omega_h} \lesssim \delta_h^2 \| D_{\boldsymbol{n}_h}^2 u^E \|_{L^2(\partial \Omega_t)} \| \varphi \|_{H^2(\Omega)}
\end{aligned}
$$

where we used that $\| \zeta^E \|_{\partial \Omega_h} = \| \boldsymbol{n}_h \cdot \nabla \varphi^E \|_{\partial \Omega_h} \lesssim \| \varphi \|_{H^2(\Omega)}$. Collecting the above bounds and using (2.3) we conclude

$$
\| e \|_{\Omega_h} \lesssim \left(h \, ||| \, (e, \eta) \, ||| + h^{\frac{1}{2}} \delta_h^{l+1} \| D_{\boldsymbol{n}_h}^l (f^E + \Delta u^E) \|_{L^2(\partial \Omega_t)} + \delta_h^2 \| D_{\boldsymbol{n}_h}^2 u^E \|_{L^2(\partial \Omega_t)} \right) \| e \|_{\Omega_h}.
$$

This ends the proof. $\square$

Corollary 3.2 *Under the same assumptions as Theorem* 3.2, *assume that* $u \in H^{k+1}(\Omega) \cap H^{\frac{7}{2}}(\Omega)$ *and* $\delta_h = O(h^{\frac{k+1}{2}})$ *then there holds*

$$
\| u^E - u_h \|_{\Omega_h} \lesssim h^{k+1} \left(|u|_{H^{k+1}(\Omega)} + \| u \|_{H^{\frac{7}{2}}(\Omega)} \right).
$$

Proof Recall from the proof of Corollary 3.1 that

$$
\| D_{\boldsymbol{n}_h}^2 u^E \|_{L^2(\partial \Omega_t)} \lesssim \| u \|_{H^{\frac{5}{2}}(\Omega)}.
$$

For $l = 1$ the following bound holds by the same argument

$$
\| D_{\boldsymbol{n}_h}(f^E + \Delta u^E) \|_{L^2(\partial \Omega_t)} \lesssim \| u \|_{H^{\frac{7}{2}}(\Omega)}.
$$

It follows from Theorem 3.1 that, with $l = 1$,

$$
\| u^E - u_h \|_{\Omega_h} \lesssim h^{k+1} |u|_{H^{k+1}(\Omega)} + h^{\frac{1}{2}} \delta_h^2 \| u \|_{H^{\frac{7}{2}}(\Omega)} + \delta_h^2 \| u \|_{H^{\frac{5}{2}}(\Omega)}.
$$

We conclude by observing that since $\delta_h = O(h^{\frac{k+1}{2}})$,

$$
h^{\frac{1}{2}} \delta_h^2 \| u \|_{H^{\frac{7}{2}}(\Omega)} + \delta_h^2 \| u \|_{H^{\frac{5}{2}}(\Omega)} \lesssim h^{k+1} \| u \|_{H^{\frac{7}{2}}(\Omega)}.
$$

$\square$

Fig. 1 Elevation of the discrete solution on triangles
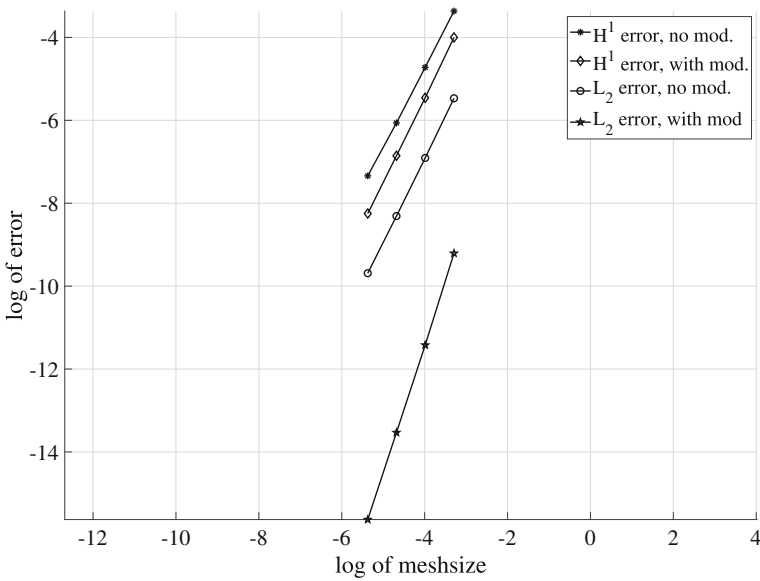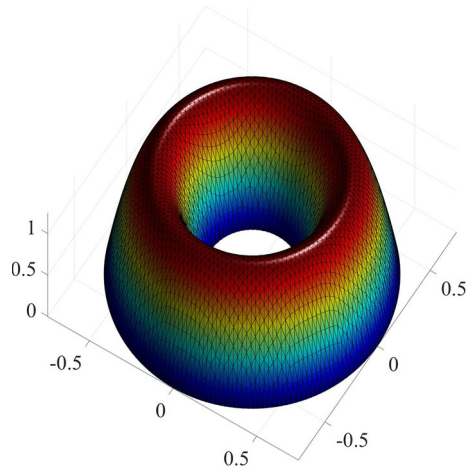

Fig. 2 Errors with and without boundary modification, P^2 case

Remark 3.3 It follows that for piecewise affine approximation the geometry error can be $O(h)$ without loss of convergence if the solution is sufficiently smooth. For quadratic approximation we need $O(h^{\frac{3}{2}})$ and for cubic we need the best approximation possible with a piecewise affine approximation of the geometry, $O(h^2)$.

Table 1 H^1 convergence for second order elements

NNO	H^1 error $\times\, 10^3$, no mod.	Rate	H^1 error $\times\, 10^3$	Rate
724	34.57	–	18.32	–
2906	8.90	1.95	4.27	2.09
11,616	2.33	1.93	1.05	2.02
46,430	0.65	1.84	0.26	2.01

Table 2 L_2 convergence for second order elements

NNO	L_2 error $\times\, 10^5$, no mod.	Rate	L_2 error $\times\, 10^5$	Rate
724	421.80	–	10.02	–
2906	100.04	2.07	4.27	3.19
11,616	24.61	2.02	1.05	3.04
46,430	6.20	1.99	0.26	3.03
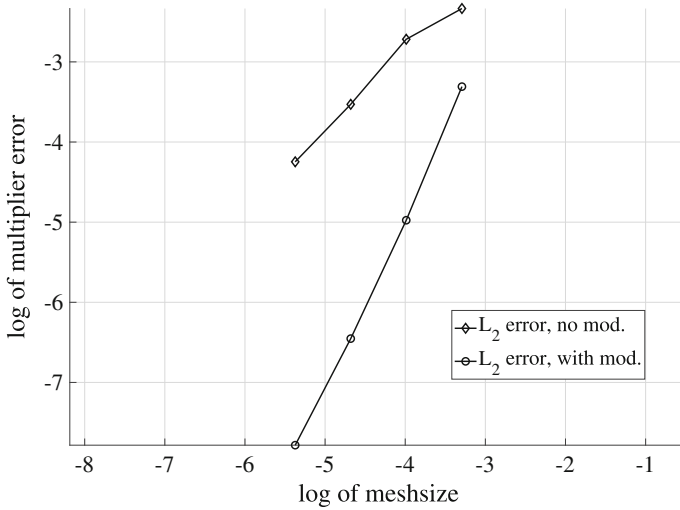
Fig. 3 Errors in the multiplier with and without boundary modification, P^2 case

Table 3 Multiplier convergence for second order elements

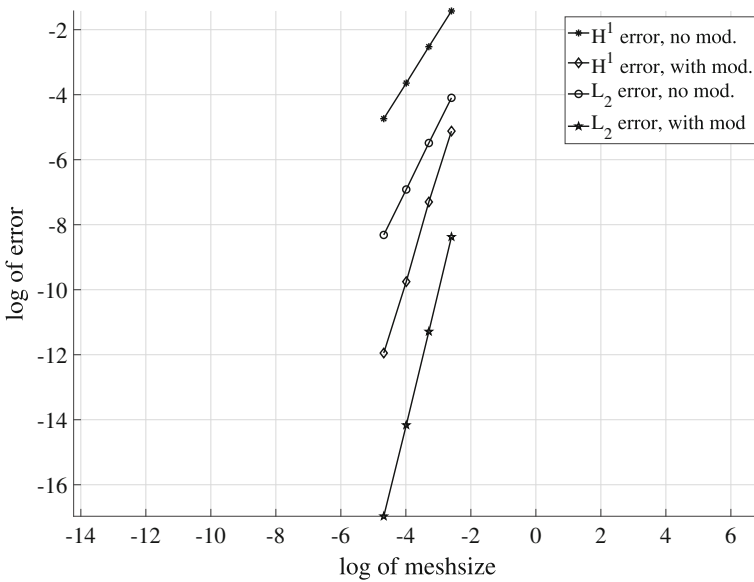| NNO	L_2 error $\times\, 10^5$, no mod.	Rate	L_2 error $\times\, 10^5$	Rate
724	96.89	–	36.55	–
2906	66.03	0.55	6.90	2.40
11,616	29.32	1.17	1.57	2.13
46,430	14.31	1.04	0.42	1.92

Fig. 4 Errors with and without boundary modification, P^3 case

Table 4 H^1 convergence for third order elements

NNO	H^1 error $\times 10^2$, no mod.	Rate	H^1 error $\times 10^4$	Rate
180	24.07	–	59.75	–
724	8.01	1.58	6.75	3.13
2906	2.62	1.61	0.58	3.53
11,616	0.88	1.58	0.06	3.17

Table 5 L_2 convergence for third order elements

NNO	L_2 error $\times 10^4$, no mod.	Rate	L_2 error $\times 10^6$	Rate
180	166.31	–	230.90	–
724	41.43	2.00	12.54	4.19
2906	9.91	2.06	0.70	4.14
11,616	2.45	2.02	0.04	4.05

4 Remarks on cut finite element methods

In the context of cut finite element methods the discontinuous multiplier spaces used above can no longer be expected to be stable. It is possible to stabilise the multiplier using Barbosa-Hughes stabilisation. However, fluctuation based multipliers are unlikely to be suitable in this context since the weak consistency of the fluctuations of the multiplier between elements depends on the geometry approximation through the
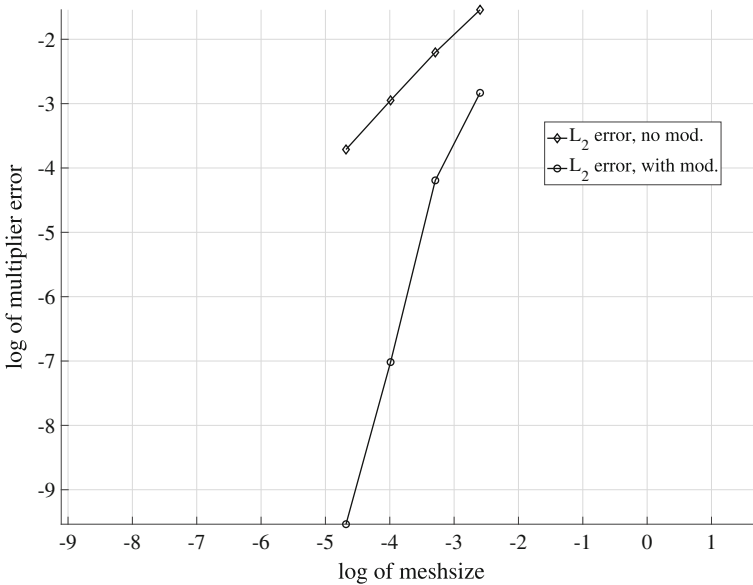
Fig. 5 Errors in the multiplier with and without boundary modification, P^3 case

Table 6 Multiplier convergence for third order elements

NNO	L_2 error $\times 10^3$, no mod.	Rate	L_2 error $\times 10^4$	Rate
180	214.00	–	588.83	–
724	110.38	0.95	151.03	1.96
2906	52.38	1.07	8.97	4.06
11,616	24.41	1.10	0.72	3.65

interface normal. Since the method is of interest when the geometry approximation is of relatively low order, this limits the possibility to use fluctuation based stabilisation.

5 Numerical examples

We show examples of higher order triangular elements with linearly interpolated boundary and low order rectangular elements with staircase boundary, using discontinuous multiplier spaces. In all examples we define the meshsize $h = 1/\sqrt{\text{NNO}}$, where NNO corresponds to the number of nodes of the lowest order FEM on the mesh in question (bilinear or affine).

5.1 Triangular elements

We first consider the case of affine triangulations of a ring $1/4 \leq r \leq 3/4$, $r = \sqrt{x^2 + y^2}$. We use the manufactured solution $u = (r - 1/4)(3/4 - r)$ and compute
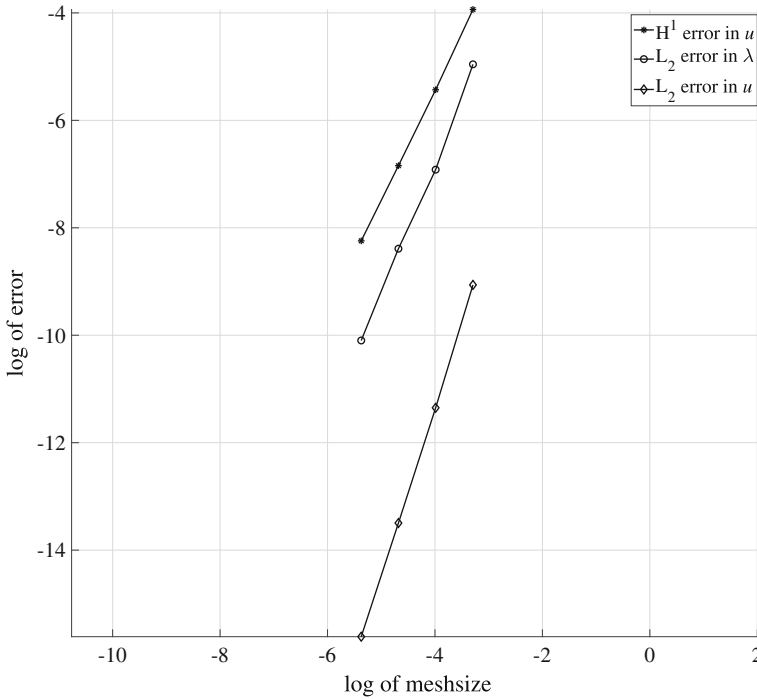
Fig. 6 Error plots for the unstable triangular element example

Table 7 Convergence for unstable pairing

NNO	H^1 error $\times 10^3$	Rate	L_2 error $\times 10^6$	Rate	Multipler error $\times 10^4$	Rate
724	19.53	–	115.86	–	70.34	–
2906	4.38	2.15	11.76	3.29	9.90	2.82
11,616	1.07	2.04	1.38	3.10	2.27	2.12
46,430	0.26	2.02	0.17	3.05	0.41	2.46

the corresponding right-hand side analytically. An elevation of the a typical discrete solution is given in Fig. 1.

We use continuous piecewise P^k polynomials, $k = 2, 3$ for the approximation of u, and for the approximation of λ we use piecewise P^{k-1} polynomials, discontinous on each element edge on $\partial \Omega_h$. To ensure *inf-sup* stability, we add hierarchical P^{k+1} bubbles on each edge in the approximation of u.

Second order elements In Fig. 2 and Tables 1 and 2 we show the convergence in $H^1(\Omega_h)$ and $L_2(\Omega_h)$ with and without boundary modification. In Fig. 3 and Table 3 we show the error in multiplier computed as $\|(-\boldsymbol{n} \cdot \nabla u)|_{\partial \Omega_h} - \lambda_h\|_{\partial \Omega_h}$. Optimal order
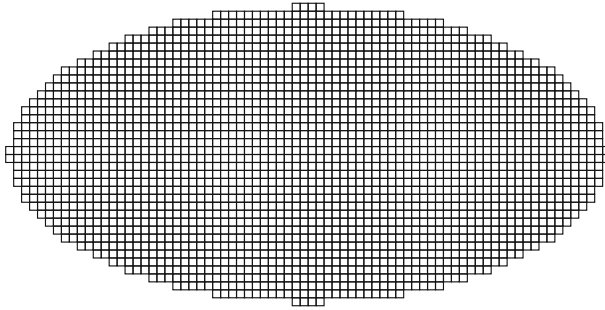
Fig. 7 A coarse mesh inside the elliptical domain

convergence is observed for the modified method, convergence $O(h^3)$ in $L_2(\Omega_h)$ and $O(h^2)$ in $H^1(\Omega_h)$; the multiplier error is approximately $O(h^2)$.

Third order elements Next we use continuous piecewise third order polynomials for the approximation of u, and for the approximation of λ we use piecewise quadratic polynomials, discontinous on each element edge on $\partial\Omega_h$. In Fig. 4 and Tables 4 and 5 we show the convergence in $H^1(\Omega_h)$ and $L_2(\Omega_h)$ with and without boundary modification. In Fig. 5 and Table 6 we show the error in multiplier computed as above. Optimal order convergence is again observed for the modified method, convergence $O(h^4)$ in $L_2(\Omega_h)$ and $O(h^3)$ in $H^1(\Omega_h)$; the multiplier error is approximately $O(h^3)$. Note that no improvement over P^2 approximations can be seen in the unmodified method due to the geometry error being dominant.

An unstable pairing of spaces We finally make the observation that our modification has a stabilising influence on the approximation. We try continuous P^2 approximations of u and discontinuous P^2 approximations of λ. In this case we get no convergence without the modification due to the violation of the *inf-sup* condition, whereas with modification we obtain the optimal convergence pattern in u and a stable multiplier convergence given in Fig. 6 and Table 7.

5.2 Rectangular elements

This example shows that it is possible to achieve optimal convergence even on a staircase boundary. We use a continuous piecewise Q_1 approximation on the (affine) rectangles, again enhanced for *inf-sup*, now by hierarchical P^2 bubble function on the boundary edges, together with edgewise constant multipliers on $\partial\Omega_h$. We use the manufactured solution $u = \sin(x^3)\cos(8y^3)$ on the domain inside the ellipse $x^2/4 + y^2 = 1$. Our computational grids consist of elements completely inside this ellipse; a typical coarse grid is shown if Fig. 7 where we note the staircase boundary. In Fig. 8 we show elevations of the numerical solutions on a finer grid without and with boundary correction. In Fig. 9 and Tables 8 and 9 we show the errors of the
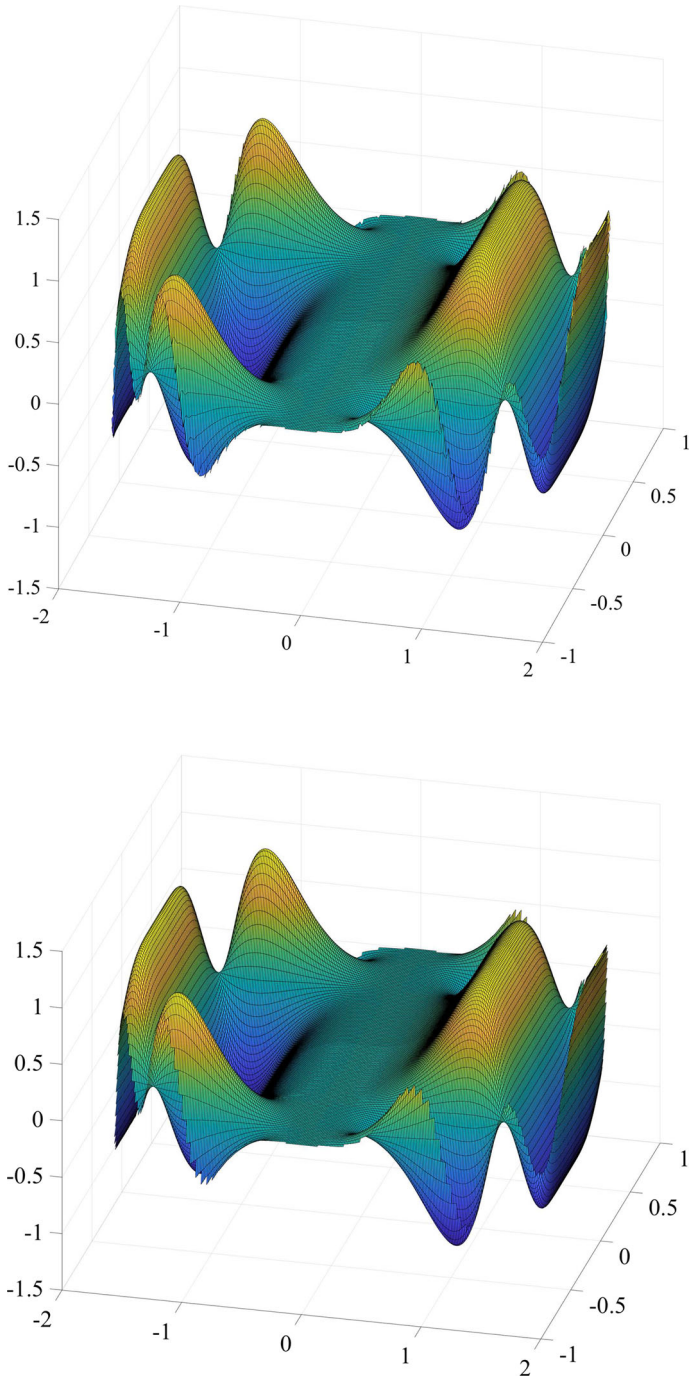
Fig. 8 Elevation of the discrete solution on rectangles for the unmodified (top) and for the modified (bottom) schemes
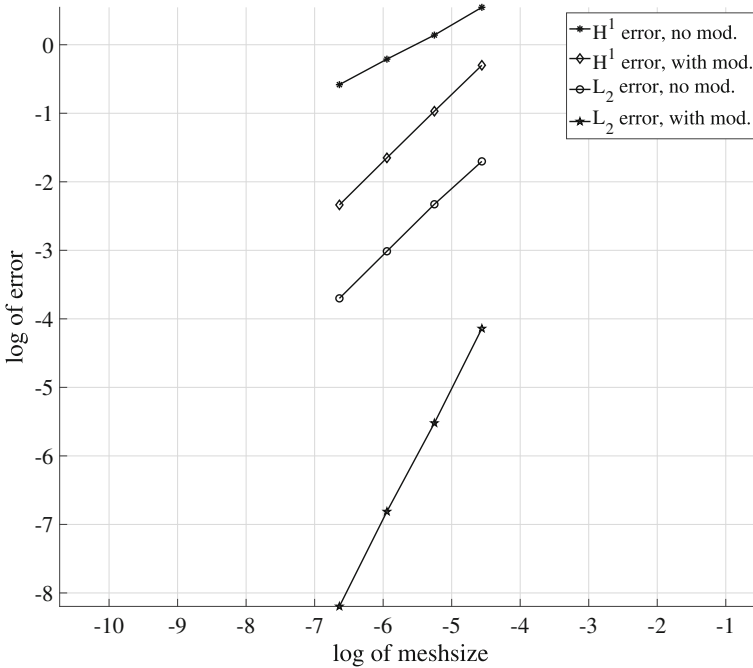
Fig. 9 Error plots for the rectangular element example

Table 8 H^1 convergence for bilinear elements

Number of nodes	H^1 error $\times 10^2$, no mod.	Rate	H^1 error $\times 10^2$	Rate
9129	172.59	–	74.10	–
36,485	115.11	0.58	37.91	0.97
145,873	81.01	0.51	19.19	0.98
583,547	55.85	0.54	9.66	0.99

Table 9 L_2 convergence for bilinear elements

Number of nodes	L_2 error $\times 10^2$, no mod.	Rate	L_2 error $\times 10^3$	Rate
9129	18.23	–	15.90	–
36,485	9.75	0.90	4.00	1.99
145,873	4.91	0.99	1.10	1.86
583,547	2.47	0.99	0.28	2.00

unmodified and modified methods. Again we observe optimal order convergence for the modified method, $O(h^2)$ in $L_2(\Omega_h)$ and $O(h)$ in $H^1(\Omega_h)$.

6 Concluding remarks

We have introduced a symmetric modification of the Lagrange multiplier approach to satisfying Dirichlet boundary conditions for Poisson's equation. This novel approach allows for affine approximations of the boundary, and thus affine elements, up to polynomial approximation order 3 without loss of convergence rate as compared to higher order boundary fitted meshes. The modification is easy to implement and only requires that the distance to the exact boundary in the direction of the discrete normal can be easily computed. In fact, the modification stabilises the multiplier method so that unstable pairs of spaces can be used, as long as there is a uniform distance to the boundary.

References

1. Bramble, J.H., Dupont, T., Thomée, V.: Projection methods for Dirichlet's problem in approximating polygonal domains with boundary-value corrections. Math. Comput. **26**, 869–879 (1972)
2. Dupont, T.: L_2 error estimates for projection methods for parabolic equations in approximating domains. In: de Boor, C. (ed.) Mathematical Aspects of Finite Elements in Partial Differential Equations, pp. 313–352. Academic Press, New York (1974)
3. Burman, E., Hansbo, P., Larson, M.G.: A cut finite element method with boundary value correction. Math. Comput. **87**(310), 633–657 (2018)
4. Burman, E., Hansbo, P., Larson, M.G.: A cut finite element method with boundary value correction for the incompressible Stokes' equations. In: Radu, F.A., Kumar, K., Berre, I., Nordbotten, J.M., Pop, I.S. (eds.) Numerical Mathematics and Advanced Applications ENUMATH 2017, pp. 183–192. Springer, Cham (2019)
5. Boiveau, T., Burman, E., Claus, S., Larson, M.: Fictitious domain method with boundary value correction using penalty-free Nitsche method. J. Numer. Math. **26**(2), 77–95 (2018)
6. Main, A., Scovazzi, G.: The shifted boundary method for embedded domain computations. Part I: poisson and Stokes problems. J. Comput. Phys. **372**, 972–995 (2018)
7. Main, A., Scovazzi, G.: The shifted boundary method for embedded domain computations. Part II: linear advection–diffusion and incompressible Navier–Stokes equations. J. Comput. Phys. **372**, 996–1026 (2018)
8. Stenberg, R.: On some techniques for approximating boundary conditions in the finite element method. J. Comput. Appl. Math. **63**(1–3), 139–148 (1995)
9. Johansson, A., Larson, M.G.: A high order discontinuous Galerkin Nitsche method for elliptic problems with fictitious boundary. Numer. Math. **123**(4), 607–628 (2013)
10. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Classics in Mathematics. Springer, Berlin (2001). Reprint of the 1998 edition
11. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series, vol. 30. Princeton University Press, Princeton (1970)
12. Burman, E., Hansbo, P.: Deriving robust unfitted finite element methods from augmented Lagrangian formulations. In: Bordas, S.P.A., Burman, E., Larson, M.G., Olshanskii, M.A. (eds.) Geometrically Unfitted Finite Element Methods and Applications, pp. 1–24. Springer, Cham (2017)
13. Ben Belgacem, F., Maday, Y.: The mortar element method for three-dimensional finite elements. RAIRO Modél. Math. Anal. Numér. **31**(2), 289–302 (1997)

14. Braess, D., Dahmen, W.: Stability estimates of the mortar finite element method for 3-dimensional problems. East–West J. Numer. Math. **6**(4), 249–263 (1998)
15. Kim, C., Lazarov, R.D., Pasciak, J.E., Vassilevski, P.S.: Multiplier spaces for the mortar finite element method in three dimensions. SIAM J. Numer. Anal. **39**(2), 519–538 (2001)
16. Ern, A., Guermond, J.-L.: Finite element quasi-interpolation and best approximation. ESAIM Math. Model. Numer. Anal. **51**(4), 1367–1385 (2017)