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HAMILTONIAN FLUID MECHANICS

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INTRODUCTION

This paper reviews the relatively recent application of the methods of Hamiltonian mechanics to problems in fluid dynamics. By *Hamiltonian mechanics* I mean all of what is often called *classical mechanics*—the subject of the textbooks by Lanczos (1970), Goldstein (1980), and Arnol'd (1978). Since the advent of quantum mechanics, Hamiltonian methods have played an increasingly important role in both the classical and quantum mechanics of particles and fields. By comparison, the introduction of Hamiltonian methods into fluid mechanics has been tardy. Why is this so?

In general mechanical systems, the Lagrangian or Hamiltonian equations of motion are *coupled* equations governing the locations and velocities of massive particles or rigid bodies. These coupled equations cannot generally be solved for any subset of the dependent variables without also finding all of the other dependent variables. By contrast, the conventional Eulerian fluid equations are *closed* equations in the velocity, density, and entropy (regarding pressure as a prescribed function of the density and entropy) that can (in principle) be solved without also finding the trajectory of every fluid particle. Once the velocity field is known, the particle trajectories can always be reconstructed by solving the equations for three independent, passively advected tracers (such as the initial Cartesian components), but these extra computations are not required if only the Eulerian fields are sought. In the special case of constant-density flow, the Eulerian equations are dramatically simpler than the general Lagrangian or Hamiltonian equations for the fluid.

From the Hamiltonian perspective, the extraordinary simplicity of the Eulerian description derives from a symmetry property of the fluid

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Hamiltonian, which is not affected by a relabeling of fluid particles with the same density and entropy. The particle-relabeling symmetry property motivates the transformation to Eulerian variables, but the Eulerian variables turn out to be noncanonical. This fact and the heavy emphasis on canonical variables in the older literature are the probable reasons why the Hamiltonian formulation of fluid mechanics went so long unexplored. The relationship between Hamiltonian mechanics and the Eulerian fluid equations is now well understood, thanks primarily to a relatively new (post–nineteenth century!) *geometric* view of mechanics that sharply distinguishes between the existence of Hamiltonian structure and the use of canonical variables. From the geometric viewpoint, the statement that *noncanonical (e.g. Eulerian) variables are sometimes useful even though the underlying dynamics is Hamiltonian* is closely analogous to the more obvious statement that *non-Cartesian (e.g. spherical) coordinates are sometimes useful even though the underlying geometry is Euclidean.*

The particle-relabeling symmetry property corresponds, by Noether's theorem, to a conservation law that turns out to be the most general statement of vorticity conservation. All of the well-known vorticity theorems are direct consequences of this law. Thus the two most distinctive characteristics of fluid mechanics—the *existence* of an abridged Eulerian description, and the central role played by vorticity (or the lack of it)—can be elegantly traced to a common origin. However, the general vorticity law cannot be stated without referring to the locations of marked fluid particles. This is but one of several important examples in which the greatest simplicity and generality are achieved only by considering the complete set of Lagrangian fluid variables. These examples suggest that the primitive picture of a fluid as a continuous distribution of massive particles is in some sense the more fundamental, and that the simplicity of the conventional Eulerian description has been purchased at a definite price.

If Hamiltonian methods merely offered a new perspective on familiar results, they would deserve little attention. However, evidence accumulates that the methods of classical mechanics comprise a powerful tool in fluid mechanics. Asymptotic approximations, conservation laws, stability theorems, and useful variable transformations all acquire a transparency and motivation that is often lacking when the corresponding manipulations are applied directly to the Eulerian equations of motion.

The utility of Hamiltonian methods seems to originate from several factors. First, Hamilton's principle is a remarkably succinct statement of dynamics, and this succinctness is itself a source of economy, as when an asymptotic expansion is substituted into the Lagrangian functional and terms are cancelled *before* Hamilton's principle is invoked to obtain the approximate equations. Second, there exists a well-known connection between the symmetry properties of the Hamiltonian and the conservation laws of the corresponding dynamical equations. This connection makes it easy to construct approximations that conserve analogues of the exact constants of motion. Similarly, it is often easy to identify new conservation laws (like conservation of wave action) that arise from approximations. Third, Hamiltonian methods are not tied to a particular choice of coordinates. In Hamiltonian perturbation theory, for example, dynamical approximations are always conjoined to transformations of the dependent variables. The freedom to simultaneously adjust both the physics and the variables used to describe it leads to final equations of maximal simplicity.

However, despite all of the above, the existence of a Hamiltonian structure is, *by itself*, meaningless because *any* set of evolution equations can be written in canonical form. To appreciate this point, consider the heat equation

$$T_t = \kappa T_{xx} \tag{0.1}$$

with periodic boundary conditions

$$T(x+2\pi, t) = T(x, t). \tag{0.2}$$

By anyone's definition, (0.1) is non-Hamiltonian. But (0.1) is the Euler-Lagrange equation corresponding to

$$\delta \iint dt\ dx[\alpha T_t + \kappa T_x \alpha_x] = 0 \tag{0.3}$$

for variations $\delta T(x, t)$ and $\delta\alpha(x, t)$. The conjugate variable α obeys the adjoint equation

$$\alpha_t = -\kappa\alpha_{xx}. \tag{0.4}$$

The construction (0.3) is open to criticism in that α is an "artificial variable," or that

$$\int dx\ \kappa T_x \alpha_x \tag{0.5}$$

is not the "physical energy." However, these objections are semantic, and variational principles like (0.3) have been seriously proposed as a way to handle dissipative systems. Moreover, the recent literature on Hamiltonian fluid mechanics contains variables that are *from a certain viewpoint* as artificial as α, and new conserved quantities that are surely as silly as (0.5). Without venturing further, let us simply agree that, while the beauty of Hamiltonian theory may reside in its formal mathematical structure, its real importance depends very much on the physical meaning attached to the mathematical symbols.

FUNDAMENTALS

1. *The Geometric View of Mechanics*

The geometric approach to classical mechanics is the subject of an increasing number of monographs, including Abraham & Marsden (1978), Arnol'd (1978), and Dubrovin et al. (1984). The following discussion is patterned after the short review by Greene (1982).

To establish essential ideas, we consider first a mechanical system composed of N discrete particles. Let m_i and $\mathbf{x}_i(\tau)$ be the mass and Cartesian location of the ith particle at time τ. Let $V(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ be the potential energy of the system. The Lagrangian is

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \sum_i 1/2 m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i - V(\mathbf{x}_1, \ldots, \mathbf{x}_N), \tag{1.1}$$

and the dynamical equations result from Hamilton's principle in the form

$$\delta \int L \, d\tau = 0, \tag{1.2}$$

where δ corresponds to arbitrary variations $\delta\mathbf{x}_i(\tau)$ in the particle trajectories, with $\delta\mathbf{x}_i = 0$ at the endpoints in τ. Alternatively, we can define the conjugate momenta

$$\mathbf{p}_i = \partial L / \partial \dot{\mathbf{x}}_i \tag{1.3}$$

and invoke Hamilton's principle in the modified form

$$\delta \int d\tau \left\{ \sum_i \mathbf{p}_i \cdot \dot{\mathbf{x}}_i - H \right\} = 0, \tag{1.4}$$

where

$$H(\mathbf{p}, \mathbf{x}) = \sum_i \mathbf{p}_i \cdot \dot{\mathbf{x}}_i - L(\mathbf{p}, \mathbf{x}) \tag{1.5}$$

is the Hamiltonian and δ now stands for arbitrary independent variations $\delta\mathbf{p}_i(\tau)$, $\delta\mathbf{x}_i(\tau)$ in the momenta and locations of the particles. The variation (1.4) yields the canonical equations,

$$\dot{\mathbf{x}}_i = \partial H / \partial \mathbf{p}_i, \qquad \dot{\mathbf{p}}_i = -\partial H / \partial \mathbf{x}_i. \tag{1.6}$$

Now define the *Poisson bracket*

$$\{F, G\} = \sum_i (\partial F/\partial \mathbf{x}_i \cdot \partial G/\partial \mathbf{p}_i - \partial F/\partial \mathbf{p}_i \cdot \partial G/\partial \mathbf{x}_i) \tag{1.7}$$

for any two functions $F(\mathbf{p}, \mathbf{x})$ and $G(\mathbf{p}, \mathbf{x})$. Then Equations (1.6) imply that

$$d/dt\ F(\mathbf{p}, \mathbf{x}) = \{F, H\}. \tag{1.8}$$

It follows from (1.7) that the Poisson bracket is antisymmetric,

$$\{F, G\} = -\{G, F\}, \tag{1.9}$$

obeys the Jacobi identity,

$$\{\{E, F\}, G\} + \{\{F, G\}, E\} + \{\{G, E\}, F\} = 0 \tag{1.10}$$

(for any E, F, G), and is nonsingular in the sense that if $\{F, G\} = 0$ for *any* choice of G, then $F = \text{constant}$.

The variational principle (1.4), the ray equations (1.6), and the statement (1.8) containing the Poisson bracket are alternative beginning points for an axiomatic theory of classical mechanics. In the older literature, the variational principle was usually regarded as the fundamental statement. However, modern treatments favor (1.8) for reasons that are next explained.

Let

$$(z^1, \ldots, z^M) = (\mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N) \tag{1.11}$$

with $M = 6N$ and note that each z^i is one Cartesian component of the location or momentum of one particular particle. In a new notation, (1.7) becomes

$$\{F, G\} = \partial F/\partial z^i J^{ij}\ \partial G/\partial z^j, \tag{1.12}$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1.13}$$

I is the $3N$-dimensional unit matrix, and repeated indices are summed. The significance of (1.12) is that (1.8–1.10) are all covariant under arbitrary transformations of the phase coordinates z^i. That is, if

$$\bar{z}^i = \bar{z}^i(z) \tag{1.14}$$

are new coordinates, $\bar{F}(\bar{z})$ is the function $F(z)$ expressed in the new coordinates, and

$$\bar{J}^{ij} = \partial \bar{z}^i/\partial z^m J^{mn}\ \partial \bar{z}^j/\partial z^n \tag{1.15}$$

transforms as a rank-two contravariant tensor, then (1.8–1.10, 1.12) hold with all dependent and independent variables v replaced by $\bar{v}$. The properties (1.9, 1.10) can also be written in terms of J^{ij} in the covariant forms

$$J^{ij} = -J^{ji} \tag{1.16}$$

and

$$J^{im}\, \partial J^{jk}/\partial z^m + J^{jm}\, \partial J^{ki}/\partial z^m + J^{km}\, \partial J^{ij}/\partial z^m = 0. \tag{1.17}$$

The complete specification of a Hamiltonian system is therefore equivalent to the choice of a scalar Hamiltonian $H(z)$ and a contravariant tensor $J^{ij}(z)$ with the properties (1.16, 1.17). All the general results of classical mechanics then follow from (1.8, 1.12, 1.16, 1.17) as an exercise in tensor analysis, in which the special properties (1.16, 1.17) of J^{ij} play a critical role. The theory of skew-symmetric tensors is very special and is most naturally stated in the language of differential forms (e.g. Arnol'd 1978). However, the present review uses nothing beyond "old-fashioned" tensor analysis.

If the coordinates z^i are canonical, then J^{ij} takes the special form (1.13). However, by Darboux's theorem any nonsingular J^{ij} with the properties (1.16, 1.17) can be brought into the form (1.13) by a transformation to canonical coordinates. For such a J^{ij}, there are in fact infinitely many sets of canonical coordinates, interrelated by canonical transformations.

There exists a close and illuminating analogy between the geometry of J^{ij} (called symplectic geometry) and ordinary Euclidean geometry with metric tensor g^{ij}. The condition (1.17) is analogous to the vanishing of the curvature tensor. Any nonsingular symmetric g^{ij} with zero curvature can be brought into the special form

$$g = I \tag{1.18}$$

by a transformation to Cartesian coordinates. Equation (1.18) is analogous to (1.13). There are infinitely many sets of Cartesian coordinates, interrelated by unitary transformations. Thus canonical transformations in phase space are the analogues of rigid rotations in ordinary Euclidean space. Just as any non-Euclidean manifold can be made Euclidean by embedding it in a higher dimensional space, any non-Hamiltonian dynamics can be made Hamiltonian by introducing auxiliary variables, as in example (0.3). However, these facts evidently do not diminish the importance of either Euclidean geometry or Hamiltonian mechanics.

It can happen that J^{ij} is singular but still satisfies (1.16) and (1.17). This situation typically occurs after a transformation from canonical coordinates to a *reduced* set of fewer coordinates, as in the transformation from Lagrangian to Eulerian fluid variables. In the reduced phase space, canonical coordinates do not exist, but all the results of (1.8, 1.12, 1.16, 1.17) still apply. If J^{ij} is singular with corank K, then it follows from the Frobenius theorem and the property (1.17) that there exist K independent functions $C_k(z)$, called Casimirs, for which $\{C_k, F\} = 0$ for *any* $F(z)$, and therefore $dC_k/dt = 0$ for *any* Hamiltonian $H(z)$. The reduced equations are complete, provided that J^{ij} depends only on the noncanonical variables. This can occur if the Hamiltonian, written in the original canonical coordinates, has symmetry properties that permit the reduction. Then the Casimirs $C_k(z)$ are the conserved quantities corresponding to these symmetries. For a lucid discussion of singular Poisson tensors and Casimirs, refer to Littlejohn (1982).

2. *The Particle-Mechanics Form of Hamilton's Principle*

The simplest form of Hamilton's principle for a perfect fluid is a straightforward generalization of (1.1) to the case of marked particles distributed *continuously* in space. This *particle-mechanics* version of Hamilton's principle was given by Herivel (1955) for the special case of incompressible flow, and by Serrin (1959) and Eckart (1960) for general compressible, nonhomentropic flow. For earlier references, see Truesdell & Toupin (1960, pp. 603–5).

Let $\mathbf{x}(a, b, c, \tau)$ be the location of the fluid particle identified by curvilinear labeling coordinates $\mathbf{a} = (a, b, c) = (a_1, a_2, a_3)$ at time τ. The labeling coordinates remain constant following the motion of the fluid particles, and they are analogous to the subscript i in Section 1. It is convenient to assign these labeling coordinates so that

$$d(\text{mass}) = da\, db\, dc. \tag{2.1}$$

Then

$$\rho = \partial(a, b, c)/\partial(x, y, z) = \partial(\mathbf{a})/\partial(\mathbf{x}) \tag{2.2}$$

is the mass-density of the fluid. Since the labeling coordinates follow the motion, (2.2) holds at all times. A direct application of $\partial/\partial\tau$ to (2.2) then yields

$$\partial\rho/\partial\tau + \rho\nabla\cdot\mathbf{u} = 0, \tag{2.3}$$

where

$$\mathbf{u} = (u, v, w) = \partial\mathbf{x}/\partial\tau \tag{2.4}$$

and

$$\nabla = (\partial_x, \partial_y, \partial_z) \tag{2.5}$$

is the gradient operator in $\mathbf{x}$-space. Thus mass conservation is implicit in the labeling of coordinates. Note that $\partial/\partial\tau$ is the same as D/Dt in conventional notation.

The Lagrangian analogous to (1.1) is

$$L = \iiint d\mathbf{a}\ \{1/2(\partial\mathbf{x}/\partial\tau)^2 - E(\partial(\mathbf{x})/\partial(\mathbf{a}), S(a,b,c)) - \Phi(\mathbf{x})\}, \tag{2.6}$$

where Φ is the potential for external forces, and the specific internal energy

$$E = E(\alpha, S) \tag{2.7}$$

is a *prescribed* thermodynamic function of the specific volume

$$\alpha = \rho^{-1} = \partial(x,y,z)/\partial(a,b,c) \tag{2.8}$$

and the specific entropy S. The entropy depends only on the labeling coordinates, in a manner determined by initial conditions. Thus

$$\partial S/\partial\tau = 0. \tag{2.9}$$

The essence of the perfect-fluid approximation is that the fluid-particle locations $\mathbf{x}(\mathbf{a}, \tau)$ enter the potential energy in (2.6) only in undifferentiated form through Φ and through the Jacobian (2.8) in E. Hamilton's principle states that

$$\delta \int L\, d\tau = 0, \tag{2.10}$$

where δ stands for arbitrary independent variations $\delta\mathbf{x}(a,b,c,\tau)$ in the particle locations. The statement (2.10) implies that

$$\delta\mathbf{x}\colon\ \partial^2\mathbf{x}/\partial\tau^2 = -\alpha\nabla p - \nabla\Phi \tag{2.11}$$

and that $p = 0$ at the (free) boundaries of the fluid, where

$$p \equiv -\partial E(\alpha, S)/\partial\alpha. \tag{2.12}$$

Note that (2.12) is the usual thermodynamic equation relating pressure and internal energy, and it may be considered the equation of state. If rigid boundaries are present, they must appear as infinite potential barriers in the function $\Phi(\mathbf{x})$. In most of what follows, we assume for convenience that the fluid is unbounded and let all variations vanish at infinite distances. However, boundaries present no essential difficulties to any of these methods.

Equations (2.3, 2.9, 2.11, 2.12) are the complete equations for a perfect fluid. The total differential of (2.7) is

$$dE = \partial E/\partial\alpha\ d\alpha + \partial E/\partial S\ dS. \tag{2.13}$$

By (2.12) and the definition

$$T \equiv \partial E/\partial S, \tag{2.14}$$

(2.13) is equivalent to

$$T\,dS = dE + p\,d\alpha. \tag{2.15}$$

All of Maxwell's thermodynamic relations follow from (2.12) and (2.14). Thus the complete dynamics and reversible thermodynamics of the perfect fluid are determined by (2.10) and the choice of $E(\alpha, S)$. Since the prescribed function $E(\alpha, S)$ is the internal energy in exact thermodynamic equilibrium, its use in the Lagrangian (2.6) for a moving fluid is really an approximation. This approximation is equivalent to the assumption of *local thermodynamic equilibrium* in the more conventional derivation of the perfect-fluid equations.

For future use, we note that the analogues of (1.3, 1.4) are

$$\mathbf{u}(\mathbf{a}, \tau) = \delta L/\delta(\partial \mathbf{x}/\partial \tau) \tag{2.16}$$

and

$$\delta \int d\tau \left\{ \iiint d\mathbf{a}\ \mathbf{u} \cdot \partial \mathbf{x}/\partial \tau - H \right\} = 0, \tag{2.17}$$

where

$$H[\mathbf{u}, \mathbf{x}] = \iiint d\mathbf{a}\ \{1/2\mathbf{u} \cdot \mathbf{u} + E + \Phi\}, \tag{2.18}$$

and δ stands for independent variations $\delta\mathbf{u}(\mathbf{a}, \tau)$ and $\delta\mathbf{x}(\mathbf{a}, \tau)$. The Poisson bracket analogous to (1.7) is

$$\{F, G\} = \iiint d\mathbf{a}\ \{\delta F/\delta \mathbf{x}(\mathbf{a}) \cdot \delta G/\delta \mathbf{u}(\mathbf{a}) - \delta F/\delta \mathbf{u}(\mathbf{a}) \cdot \delta G/\delta \mathbf{x}(\mathbf{a})\}, \tag{2.19}$$

where functional derivatives have replaced the ordinary derivatives in (1.7).

3. *Eulerian Forms of Hamilton's Principle*

The fluid motion is a time-dependent map

$$\mathbf{x} = \mathbf{x}(\mathbf{a}, \tau) \tag{3.1}$$

from $\mathbf{a}$-space into $\mathbf{x}$-space, and Hamilton's principle requires that the action (2.10) be stationary for arbitrary variations in this map. Since each forward map (3.1) uniquely determines an inverse map

$$\mathbf{a} = \mathbf{a}(\mathbf{x}, t) \tag{3.2}$$

from $\mathbf{x}$-space into $\mathbf{a}$-space, Hamilton's principle is obviously equivalent to the statement that the action be stationary for arbitrary variations $\delta\mathbf{a}(\mathbf{x}, t)$ in the inverse map (3.2). Here $t = \tau$, but (x, y, z, t) are independent coordinates. This simple interchange between dependent and independent variables leads to the various *Eulerian* forms of Hamilton's principle, in which the dependent variables are varied at fixed locations $\mathbf{x}$.

The early attempts to formulate an Eulerian version of Hamilton's principle (e.g. Clebsch 1859, Bateman 1929, Eckart 1938) were ad hoc and only partly successful in that they yielded dynamical equations whose solutions are only a subset of the solutions to the perfect-fluid equations. The first general Eulerian versions of Hamilton's principle were those of Lin (1963) and Seliger & Whitham (1968). For a thorough review of the early literature, see Finlayson (1972). Lin's key contribution was the introduction of new constraints, which, as explained by Bretherton (1970), force an equivalence between the Eulerian and particle-mechanics versions of Hamilton's principle. However, it is still widely unappreciated that these two versions are really the same principle, and the following discussion is designed to emphasize this point. When the Eulerian version is *derived* from the particle-mechanics version (as here), then Lin's constraints appear as automatic requirements and not as a step requiring extraordinary ingenuity.

By (2.6, 2.10) and the reasoning given above, Hamilton's principle must be equivalent to the statement that

$$\delta \iiiint dt\; d\mathbf{x}\; \partial(\mathbf{a})/\partial(\mathbf{x})\,\{1/2\mathbf{u}\cdot\mathbf{u} - E(\partial(\mathbf{x})/\partial(\mathbf{a}), S(\mathbf{a})) - \Phi(\mathbf{x})\} = 0 \tag{3.3}$$

for arbitrary variations $\delta\mathbf{a}(\mathbf{x}, t)$. To carry out this variation, we must express the entire integrand of (3.3) in terms of $\mathbf{a}(\mathbf{x}, t)$ and its derivatives. To express the velocity $\mathbf{u}$ as derivatives of $\mathbf{a}$, we solve the three identities

$$\partial\mathbf{a}/\partial\tau = 0 = \partial\mathbf{a}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{a} \tag{3.4}$$

for the three components of $\mathbf{u}$ and substitute the results back into (3.3). Equivalently, we can append the three equations (3.4) as constraints on (3.3) and then vary *both* $\mathbf{a}(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$. Thus Hamilton's principle is equivalent to

$$\delta \iiiint dt\; d\mathbf{x}\; \partial(\mathbf{a})/\partial(\mathbf{x})\,\{1/2\mathbf{u}\cdot\mathbf{u} - E(\partial(\mathbf{x})/\partial(\mathbf{a}), S(\mathbf{a})) - \Phi(\mathbf{x}) - \mathbf{A}\cdot D\mathbf{a}/Dt\} = 0 \tag{3.5}$$

for variations $\delta\mathbf{u}$, $\delta\mathbf{a}$, and $\delta\mathbf{A}(\mathbf{x}, t)$. Here $\mathbf{A} = (A, B, C) = (A_1, A_2, A_3)$ is the set of Lagrange multipliers corresponding to the *Lin constraints* (3.4).

The statement (3.5) is one of many forms of Hamilton's principle that use (x, y, z, t) as independent variables. These variational principles differ from one another in the integrand of the Lagrangian and in the quantities to be varied, but their mathematical equivalence is usually easy to establish. The velocity variation of (3.5) yields

$$\delta\mathbf{u}: \mathbf{u} = A\nabla a + B\nabla b + C\nabla c = A_i\nabla a_i, \tag{3.6}$$

where repeated indices are summed. Equation (3.6) can be used to completely eliminate the velocity $\mathbf{u}$ from (3.5), which, after some cancellations, becomes

$$\delta \iiiint dt\, d\mathbf{x}\, \partial(\mathbf{a})/\partial(\mathbf{x}) \{\mathbf{A}\cdot\partial\mathbf{a}/\partial t + 1/2(A_i\nabla a_i)^2 + E + \Phi\} = 0 \tag{3.7}$$

for variations $\delta\mathbf{A}$ and $\delta\mathbf{a}(\mathbf{x}, t)$. Note that ρA_i and a_i form canonically conjugate pairs in (3.7). It is straightforward to verify that the equations resulting from (3.7) are equivalent to the perfect-fluid equations.

The variational principle (3.7) is one of many obtained by Seliger & Whitham (1968) using a rather different approach. The present derivation emphasizes the close connection between (3.7) and the particle-mechanics form of Hamilton's principle, and it puts a clear physical interpretation on the "potentials" $\mathbf{a}$ and $\mathbf{A}$. The $\mathbf{a}$ are labeling coordinates that can be assigned in numerous ways to satisfy (2.2). But once the a_i have been chosen, the A_i are uniquely determined from (3.6) as the projections of $\mathbf{u}$ on the curvilinear basis vectors ∇a_i. The ∇a_i form a basis provided only that the density ρ is nonzero.

Now if either (2.2) or its time derivative (2.3) is appended as a constraint on (3.5), then we can replace $\partial(\mathbf{a})/\partial(\mathbf{x})$ by ρ and vary $\mathbf{a}(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$, *and* $\rho(\mathbf{x}, t)$ independently. Thus, we have

$$\begin{aligned}\delta \iiiint dt\, d\mathbf{x}\, \{\rho[1/2\mathbf{u}\cdot\mathbf{u} - E(\rho^{-1}, S(\mathbf{a})) - \Phi(\mathbf{x})\\ - \boldsymbol{\zeta}\cdot D\mathbf{a}/Dt] + \phi[\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u})]\} = 0\end{aligned} \tag{3.8}$$

for independent variations $\delta\rho$, $\delta\mathbf{u}$, $\delta\mathbf{a}$, $\delta\boldsymbol{\zeta}$, and $\delta\phi(\mathbf{x}, t)$. Here $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3) = (\zeta, \eta, \theta)$ and ϕ are the Lagrange multipliers corresponding to (3.4) and (2.3). The velocity variation of (3.8) yields

$$\delta\mathbf{u}: \mathbf{u} = \zeta\nabla a + \eta\nabla b + \theta\nabla c + \nabla\phi, \tag{3.9}$$

which can again be used to eliminate $\mathbf{u}$ from (3.8). After some cancellations and an integration by parts, (3.8) becomes

$$\delta \iiiint dt\, d\mathbf{x}\, \rho\{\zeta \partial a/\partial t + \eta \partial b/\partial t + \theta \partial c/\partial t + \partial \phi/\partial t$$
$$+ 1/2\mathbf{u}\cdot\mathbf{u} + E(\rho^{-1}, S(\mathbf{a})) + \Phi\} = 0 \tag{3.10}$$

for variations $\delta\mathbf{a}$, $\delta\zeta$, $\delta\rho$, and $\delta\phi(\mathbf{x}, t)$. In (3.10), $\mathbf{u}$ is simply an abbreviation for (3.9). The variations yield the equations

$$\begin{aligned} &\delta\zeta_i\text{: } Da_i/Dt = 0, \qquad \delta a_i\text{: } D\zeta_i/Dt = (\partial E/\partial S)(\partial S/\partial a_i),\\ &\delta\phi\text{: } \partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0,\\ &\delta\rho\text{: } \boldsymbol{\zeta}\cdot\partial\mathbf{a}/\partial t + \partial\phi/\partial t + 1/2\mathbf{u}\cdot\mathbf{u} + \Phi + E + P/\rho = 0.\end{aligned} \tag{3.11}$$

Now (3.11) are eight evolution equations for the eight scalar dependent variables. However, (2.17) and (3.7) both yield six evolution equations for six dependent variables. This suggests that (3.11) can be simplified with no loss in generality. First suppose that the fluid is homentropic (i.e. that the entropy is everywhere constant). As explained below, we can assign (a, b, c) and ϕ so that $\theta = 0$ in (3.9) for any initial velocity $\mathbf{u}$. Then θ remains zero by (3.11) (with $S =$ constant). This means that

$$\delta \iiiint dt\, d\mathbf{x}\, \rho\{\zeta \partial a/\partial t + \eta \partial b/\partial t + \partial \phi/\partial t + 1/2\mathbf{u}\cdot\mathbf{u} + E(\rho^{-1}) + \Phi\} = 0 \tag{3.12}$$

for variations $\delta\zeta$, δa, $\delta\eta$, δb, $\delta\rho$, and $\delta\phi(\mathbf{x}, t)$, where $\mathbf{u}$ stands for

$$\mathbf{u} = \zeta\nabla a + \eta\nabla b + \nabla\phi. \tag{3.13}$$

The variations yield (3.11) with θ and c set formally to zero. Further abridgments of (3.13) are possible, but these correspond to special solutions of the perfect-fluid equations (Section 5).

Boozer (1985) gives a transparent proof that any $\mathbf{u}(\mathbf{x})$ can be represented as in (3.13). First, by the interpretation of (a, b, c) as labeling coordinates, any $\mathbf{u}$ has the representation (3.6). The a_i and A_i are always single-valued functions of $\mathbf{x}$. Now let $\phi(\mathbf{x}(\mathbf{a}))$ be the solution to

$$\partial\phi/\partial c|_{a,b} = C(\mathbf{x}(\mathbf{a})). \tag{3.14}$$

Then (3.6) takes the form of (3.13) with

$$\zeta = A - \partial\phi/\partial a \quad \text{and} \quad \eta = B - \partial\phi/\partial b. \tag{3.15}$$

The function $\phi(a, b, c)$ (and hence ζ and η) will be single valued if the labels (a, b, c) are assigned so that there are no closed surfaces of constant a, b, or c. This is always possible. In particular, (a, b, c) can always be chosen to have the same topology as Cartesian coordinates.

For general nonhomentropic flow, it is often possible to choose the entropy itself as one of the labeling coordinates. Suppose that ∇S is nowhere initially zero. Then we can always assign (a, b, c) to satisfy both (2.2) and (say) $S = b$. By the same reasoning as above, the initial velocity has the representation (3.9) with $\theta = 0$ and $b = S$. (There are no closed surfaces of constant entropy if $\nabla S \neq 0$.) Again, (3.11) implies that $D\theta/Dt = 0$. Thus, we have

$$\delta \iiiint dt\, d\mathbf{x}\, \rho\{\zeta\partial a/\partial t + \eta\partial S/\partial t + \partial\phi/\partial t + 1/2\mathbf{u}\cdot\mathbf{u} + E(\rho^{-1}, S) + \Phi(\mathbf{x})\} = 0 \tag{3.16}$$

for variations $\delta\zeta$, δa, $\delta\eta$, δS, $\delta\rho$, and $\delta\phi(\mathbf{x}, t)$, where

$$\mathbf{u} = \zeta\nabla a + \eta\nabla S + \nabla\phi. \tag{3.17}$$

The variations yield the following equations:

$$\begin{aligned}
&\delta\zeta\colon Da/Dt = 0, \qquad \delta a\colon D\zeta/Dt = 0,\\
&\delta\eta\colon DS/Dt = 0, \qquad \delta S\colon D\eta/Dt = T,\\
&\delta\phi\colon \partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&\delta\rho\colon \zeta\partial a/\partial t + \eta\partial S/\partial t + \partial\phi/\partial t + 1/2\mathbf{u}\cdot\mathbf{u} + \Phi + E + P/\rho = 0,
\end{aligned} \tag{3.18}$$

where the temperature T is defined by (2.14).

The statement (3.16) (and slight modifications thereof) is the best-known Eulerian version of Hamilton's principle. It is straightforward to show that (3.17, 3.18) are equivalent to the perfect-fluid equations, and van Saarloos (1981) gives an explicit canonical transformation between the canonical variables in (3.16) and those in (2.17). (See also Broer & Kobussen 1974.) However, (3.16), unlike (3.7), becomes generally invalid (as explained in Section 5) if the entropy is constant over even infinitesimal volumes of the fluid. This failure occurs not for deep physical reasons, but only because a locally constant entropy cannot serve as a particle label.

The primary reason why (3.16) has achieved such popularity seems to be that it involves a minimal number of "nonphysical" dependent variables (a, ζ, η). However, I believe that the distinction between "physical" and "nonphysical" variables has been overdrawn. The "nonphysical" variables are all either particle labels or closely related thereto. These labels would acquire an indisputable physical significance if only the internal energy were allowed to depend on solute concentrations that are conserved following the fluid particles. This is true whether or not the solute concentrations have topological properties that make them suitable themselves as particle-labeling variables.

The variational principles given above are merely representative of an infinite number of possibilities. Seliger & Whitham (1968) give many more examples. Virasoro (1981) has given an interesting Hamiltonian formulation of two-dimensional incompressible flow in which the vorticity appears as the momentum coordinate. Griffa (1981) discusses the canonical transformations relating the particle-mechanics and Eulerian versions of Hamilton's principle to Virasoro's formulation.

4. *The Particle-Relabeling Symmetry*

The particle labels $\mathbf{a}(\mathbf{x}, t)$ enter the Lagrangian (2.6) only through the density $\partial(\mathbf{a})/\partial(\mathbf{x})$ and the entropy $S(a, b, c)$. Thus the potential energy terms in the Lagrangian are unaffected by particle-label variations $\delta\mathbf{a}(\mathbf{x}, t)$ that leave the density and entropy unchanged. By Noether's theorem, this symmetry property corresponds to a conservation law. The conservation law turns out to be the most general statement of vorticity conservation. The connection between the particle-relabeling symmetry property and the general vorticity conservation law has been discovered in various forms by Calkin (1963), Bretherton (1970), Friedman & Schutz (1978), Ripa (1981), Salmon (1982), Henyey (1982, 1983), and undoubtedly others.

First suppose that the fluid is homentropic. Then the particle labels enter (2.6) only through the Jacobian (2.2). Let $\delta\mathbf{a}(\mathbf{x}, t)$ be such that

$$\delta\partial(a, b, c)/\partial(x, y, z) = 0. \tag{4.1}$$

This implies that

$$\partial\delta a/\partial a + \partial\delta b/\partial b + \partial\delta c/\partial c = 0. \tag{4.2}$$

Thus

$$\delta\mathbf{a} = \nabla_a \times \mathbf{T} \tag{4.3}$$

for some $\mathbf{T}(\mathbf{a}, \tau)$, where

$$\nabla_a = (\partial_a, \partial_b, \partial_c) \tag{4.4}$$

is the gradient operator in $\mathbf{a}$-space. For such a variation, we have

$$\begin{aligned}\delta \int L\, d\tau &= \iiiint d\tau\, d\mathbf{a}\ \partial x_i/\partial\tau\ \delta(\partial x_i/\partial\tau)\\ &= -\iiiint d\tau\, d\mathbf{a}\ (\partial x_i/\partial\tau)(\partial x_i/\partial a_j)(\partial(\delta a_j)/\partial\tau)\\ &= -\iiiint d\tau\, d\mathbf{a}\ \mathbf{A}\cdot\partial(\delta\mathbf{a})/\partial\tau, \end{aligned}\tag{4.5}$$

where

$$\mathbf{A} = u\nabla_a x + v\nabla_a y + w\nabla_a z \tag{4.6}$$

is the "reciprocal" of (3.6). Substitution from (4.3) and an integration by parts yields

$$\delta \int L\, d\tau = \iiiint d\tau\, d\mathbf{a}\, \mathbf{T} \cdot \partial/\partial\tau[\nabla_a \times \mathbf{A}]. \tag{4.7}$$

But $\mathbf{T}$ is arbitrary and (4.7) must vanish by Hamilton's principle. It follows that

$$\partial/\partial\tau(\nabla_a \times \mathbf{A}) = 0. \tag{4.8}$$

The conservation law (4.8) was discovered by Eckart (1960), but he did not notice the connection with the particle-relabeling symmetry property.

Equation (4.8) is a general statement of vorticity conservation. All the well-known vorticity theorems for homentropic flow are consequences of (4.8). Let $\theta(a, b, c)$ be any quantity that is conserved on fluid particles. Then, by (4.8),

$$\partial/\partial\tau[(\nabla_a \times \mathbf{A}) \cdot \nabla_a \theta] = 0 \tag{4.9}$$

is also conserved. By (4.6), we have

$$(\nabla_a \times \mathbf{A}) \cdot \nabla_a \theta = \rho^{-1}(\nabla \times \mathbf{u}) \cdot \nabla\theta. \tag{4.10}$$

The statement

$$\partial/\partial\tau[\rho^{-1}(\nabla \times \mathbf{u}) \cdot \nabla\theta] = 0 \tag{4.11}$$

expresses the conservation of potential vorticity. Since $\theta(a, b, c)$ is an arbitrary tracer, (4.8) and (4.11) are actually equivalent.

Now consider any closed loop in $\mathbf{a}$-space. By (4.8), it follows that

$$\partial/\partial\tau \oint \mathbf{A} \cdot d\mathbf{a} = 0. \tag{4.12}$$

But

$$\mathbf{A} \cdot d\mathbf{a} = \mathbf{u} \cdot d\mathbf{x} \tag{4.13}$$

by (4.6). The statement

$$\partial/\partial\tau \oint \mathbf{u} \cdot d\mathbf{x} = 0 \tag{4.14}$$

is Kelvin's theorem. Bretherton (1970) derived (4.14) directly by considering the virtual displacement of fluid particles around a closed material loop.

Next consider any fixed volume V in $\mathbf{a}$-space with surface everywhere tangent to $\nabla_a \times \mathbf{A}$. It follows easily from (4.8) that

$$d/dt \iiint_V d\mathbf{a}\,(\nabla_a \times \mathbf{A})\cdot\mathbf{A} = 0. \tag{4.15}$$

But we also have

$$(\nabla_a \times \mathbf{A})\cdot\mathbf{A} = \rho^{-1}(\nabla \times \mathbf{u})\cdot\mathbf{u}. \tag{4.16}$$

Thus, it follows that

$$d/dt \iiint_{V'} d\mathbf{x}\,(\nabla \times \mathbf{u})\cdot\mathbf{u} = 0, \tag{4.17}$$

where V', the corresponding volume in $\mathbf{x}$-space, is a material volume of closed vortex tubes. The statement (4.17) expresses the conservation of helicity. Note that (4.9) and (4.15) follow immediately from (4.8), whereas the original derivations of (4.11) and (4.17) [by Ertel (1942) and Moffatt (1969)] required considerable ingenuity.

Now consider general nonhomentropic flow. It is again convenient to let the entropy be one of the labeling coordinates. We therefore set $c = S$ and consider particle-label variations satisfying

$$\delta\partial(a, b, S)/\partial(x, y, z) = 0 \quad \text{and} \quad \delta S = 0. \tag{4.18}$$

These variations correspond to a relabeling of fluid particles within surfaces of constant entropy. Now (4.18) implies that

$$\delta a = -\partial\psi/\partial b, \qquad \delta b = \partial\psi/\partial a \tag{4.19}$$

for some $\psi(a, b, S, \tau)$. For such variations, we have

$$\delta \int L\,d\tau = -\iiiint d\tau\,d\mathbf{a}\,\psi\,\partial/\partial\tau[\partial A/\partial b - \partial B/\partial a]. \tag{4.20}$$

Since ψ is arbitrary, it follows that

$$\partial/\partial\tau[(\nabla_a \times \mathbf{A})\cdot\nabla_a S] = 0. \tag{4.21}$$

The analogue of (4.11) is therefore

$$\partial/\partial\tau[\rho^{-1}(\nabla \times \mathbf{u})\cdot\nabla S] = 0. \tag{4.22}$$

For nonhomentropic flow, (4.14) applies only within surfaces of constant entropy.

The derivation of the general vorticity conservation law from the particle-relabeling symmetry property provides an elegant unification that is lacking in conventional derivations. For any continuum, the general vorticity law is simply that which results from the most general transformation of labeling coordinates that leaves the "physical" variables unchanged. The symmetry approach also provides an important motivation: The conservation law is known to exist as soon as an inspection of the Lagrangian reveals the symmetry property. One need not rely on unguided manipulations. Finally, the symmetry approach shows that vorticity conservation is a consequence of the continuum approximation. It has no analogue in particle mechanics, where the particle labels cannot be varied continuously.

5. *Flows With Special Symmetry*

We now consider abridgments of the Eulerian variational principle (3.9, 3.10) that are more drastic than (3.12, 3.13) or (3.16, 3.17). These further abridgments yield dynamical equations whose solutions agree with the perfect-fluid equations but have strongly constrained vorticities. These special solutions represent "unbroken symmetries" of the general solution to the perfect-fluid equations.

First suppose that the fluid is homentropic. If *both* $Db/Dt = 0$ and $Dc/Dt = 0$ are used to justify the restriction $b \equiv c \equiv$ constant in (3.9, 3.10) (with $S \equiv$ constant), the resulting variational principle is

$$\delta \iiiint dt\, d\mathbf{x}\, \rho\{\zeta \partial a/\partial t + \partial\phi/\partial t + 1/2\mathbf{u}\cdot\mathbf{u} + E(\rho^{-1}) + \Phi\} = 0 \tag{5.1}$$

for variations $\delta\zeta$, δa, $\delta\rho$, and $\delta\phi(\mathbf{x}, t)$, where

$$\mathbf{u} = \zeta\nabla a + \nabla\phi \tag{5.2}$$

is the *Clebsch representation* of the velocity $\mathbf{u}$. It is easy to verify that the equations resulting from (5.1, 5.2) have solutions that also solve the perfect-fluid equations. However, as shown by Bretherton (1970), it follows from (5.2) that these solutions all have zero helicity, and therefore they represent a special class of solutions to the perfect-fluid equations. Thus the well-known proof (e.g. Lamb 1932, Article 167) that any vector field $\mathbf{u}$ has the local representation (5.2) does not apply to volumes of arbitrary size.

If all three of the constraints $D\mathbf{a}/Dt = 0$ are used to set $\mathbf{a} \equiv$ constant (still assuming constant entropy), then (3.9, 3.10) reduces to the variational principle for irrotational flow given by Broer (1974), namely

$$\delta \iiiint dt\, d\mathbf{x}\, \rho\{\partial\phi/\partial t + 1/2\mathbf{u}\cdot\mathbf{u} + E(\rho^{-1}) + \Phi(\mathbf{x})\} = 0, \tag{5.3}$$

where

$$\mathbf{u} = \nabla\phi \tag{5.4}$$

for variations $\delta\rho$ and $\delta\phi(\mathbf{x}, t)$. In this case the vorticity is everywhere zero. A repetition of the steps leading up to (5.3, 5.4) for a flow with a free surface and a constant density yields the important variational principle discovered by Luke (1967). For a review of the Hamiltonian formulations of surface gravity waves, see Miles (1981).

Now suppose that the flow is nonhomentropic. We again assume that $\nabla S \neq 0$ and take the entropy as one of the labels ($S = b$). In this case the restriction $a \equiv$ constant in (3.16, 3.17) yields the variational principle

$$\delta \iiiint dt\, d\mathbf{x}\, \rho\{\eta\partial S/\partial t + \partial\phi/\partial t + 1/2\mathbf{u}\cdot\mathbf{u} + E(\rho^{-1}, S) + \Phi\} = 0 \tag{5.5}$$

where

$$\mathbf{u} = \eta\nabla S + \nabla\phi. \tag{5.6}$$

The solutions resulting from (5.5, 5.6) are again solutions of the general equations but always have vanishing circulation $\oint \mathbf{u}\cdot d\mathbf{x}$ in isentropic surfaces (Milder 1982). A further abridgment of the variational principle for general nonhomentropic flow is impossible because the internal energy E must retain an arbitrary entropy dependence.

The above three restricted forms of Hamilton's principle represent flows with vanishing *material* vorticity invariants. Such flows arise in initially quiescent regions into which disturbances propagate without a transfer of fluid. It is logical that these restricted flows should correspond to abridgments of Hamilton's principle in which particle labels are set formally to zero because, as shown in Section 4, the vorticity laws arise from the particle-relabeling symmetry. Section 9 shows that these ideas extend to other systems, with somewhat surprising consequences.

6. *Poisson Brackets*

The state of a perfect fluid at a fixed time τ corresponds to a point in an infinite-dimensional phase space in which each dimension represents the value of one component of $\mathbf{u}(\mathbf{a})$ or $\mathbf{x}(\mathbf{a})$ at a fixed value of $\mathbf{a}$. The six Lagrangian fields

$$\{\mathbf{u(a)}, \mathbf{x(a)}\} \tag{6.1}$$

uniquely determine the five Eulerian fields

$$\{\mathbf{u(x)}, \rho(\mathbf{x}), S(\mathbf{x})\}. \tag{6.2}$$

However, each choice of (6.2) corresponds to infinitely many choices of (6.1). Thus (6.2) corresponds to a point in a *reduced phase space* for the fluid.

Let $F = F[\mathbf{u(a)}, \mathbf{x(a)}]$ and G be arbitrary functionals of the exact state (6.1) of the fluid. Then the Poisson bracket $\{F, G\}$ is defined by (2.19). This same bracket can be written in the general form

$$\{F, G\} = \iiint d\mathbf{y}_1 \iiint d\mathbf{y}_2\ \delta F/\delta v_i(\mathbf{y}_1)\, \{v_i(\mathbf{y}_1), v_j(\mathbf{y}_2)\}\ \delta G/\delta v_j(\mathbf{y}_2), \tag{6.3}$$

where

$$\{v_i(\mathbf{y})\} \tag{6.4}$$

are a new set of six dependent and three independent variables obtained by arbitrary transformation of (6.1). The coefficient $\{v_i(\mathbf{y}_1), v_j(\mathbf{y}_2)\}$ can be calculated from (2.19). Note that (6.3) is simply the functional analogue of (1.15).

Now it can happen that F and G depend on (6.1) only through (6.2). The Hamiltonian (2.18) written as

$$H = \iiint d\mathbf{x}\ \rho\{1/2\mathbf{u}\cdot\mathbf{u} + E(\rho^{-1}, S) + \Phi\} \tag{6.5}$$

is one such *Eulerian* functional. Then $\{F, G\}$ can be expressed as (6.3), with (6.4) replaced by (6.2). A lengthy calculation of $\{\mathbf{u(x)}, \rho(\mathbf{x})\}$, etc., yields the *Eulerian* bracket discovered by Morrison & Greene (1980, 1982):

$$\begin{aligned}\{F, G\}_e = -\iiint d\mathbf{x}\ \{&[\delta F/\delta\rho\, \nabla\cdot\delta G/\delta\mathbf{u} + \delta F/\delta\mathbf{u}\cdot\nabla\,\delta G/\delta\rho] \\ &+[\rho^{-1}\nabla\times\mathbf{u}\cdot(\delta G/\delta\mathbf{u}\times\delta F/\delta\mathbf{u})] \\ &+[\rho^{-1}\nabla S\cdot(\delta F/\delta S\,\delta G/\delta\mathbf{u} - \delta G/\delta S\,\delta F/\delta\mathbf{u})]\}.\end{aligned} \tag{6.6}$$

The corresponding brackets for a variety of examples from fluid dynamics, plasma physics, and field theory are given by Morrison (1982) and Marsden & Montgomery (1986). The bracket (6.6) inherits the covariant properties (1.9, 1.10) of the canonical bracket (2.19). However, unlike (2.19), (6.6) is singular in the sense that

$$\{F, C\}_e = 0 \tag{6.7}$$

for *any* functional F, where

$$C = \iiint d\mathbf{x}\ \rho f(q), \tag{6.8}$$

and $f(q)$ is an arbitrary function of the potential vorticity

$$q = \rho^{-1}(\nabla \times \mathbf{u}) \cdot \nabla S. \tag{6.9}$$

In the terminology of Section 1, C is a Casimir functional of the singular bracket (6.6). Such Casimirs play an important role in the stability theory discussed in Section 7.

The Eulerian bracket is significant because (6.5, 6.6) and

$$dF/dt = \{F, H\}_e \tag{6.10}$$

comprise a closed Hamiltonian dynamics that contains only the Eulerian variables (6.2). No particle labels, Lin constraints, or velocity potentials are required. The singularity (6.7) reflects the projective character of the transformation from Lagrangian to Eulerian coordinates. The Casimir invariants (6.8) correspond to the symmetry property that permits this reduction. A group-theoretic picture of the reduction from Lagrangian to Eulerian fluid variables has been given by Arnol'd (1966a) and Marsden & Weinstein (1983).

SELECTED APPLICATIONS

We now turn to three broad applications of the fundamental ideas presented in Sections 1–6. The applications include nonlinear stability theory, the theory of interactions between mean flows and superposed disturbances, and the derivation of approximate dynamical equations that retain analogues of exact conservation laws. Each of these applications really deserves (and two have received) a separate review. The very limited goal of the following brief discussion is to explain the connection between these applications and the fundamentals of Hamiltonian fluid mechanics.

7. *Nonlinear Stability Theory*

The Liapunov method of stability analysis seeks a constant of motion that is an extremum at the state whose stability is in question. Arnol'd (1966b, 1969) introduced a powerful version of the Liapunov method based upon the Casimir invariants of Hamiltonian theory. Arnol'd's method has since been applied to a great many hydrodynamical stability problems. For extensive reviews, see Holm et al. (1985) and Abarbanel et al. (1986). Here we focus on the relationship of Arnol'd's method to the ideas developed in Sections 1–6.

Consider first a mechanical system with discrete degrees of freedom in the notation of Section 1. If the system is steady in coordinates z^i at the equilibrium point $\mathbf{z}_{eq}$, then by (1.8, 1.12) we have

$$0 = dz_i/dt = J^{ij}\partial H/\partial z^j \quad \text{at} \quad \mathbf{z} = \mathbf{z}_{eq}. \tag{7.1}$$

The summation convention is in effect. If J^{ij} is nonsingular, then (7.1) implies that

$$\partial H/\partial z^j = 0 \quad \text{at} \quad \mathbf{z} = \mathbf{z}_{eq}. \tag{7.2}$$

If, however, J^{ij} is singular with corank K, then, as stated in Section 1, there exist K independent Casimir functions $C_k(\mathbf{z})$ such that

$$J^{ij}\partial C_k/\partial z^j = 0 \quad \text{at every } \mathbf{z}. \tag{7.3}$$

In this case, (7.1) implies only that

$$\partial H/\partial z^j + \lambda_k \partial C_k/\partial z^j = 0 \quad \text{at} \quad \mathbf{z} = \mathbf{z}_{eq} \tag{7.4}$$

for K constants $\{\lambda_k\}$. In other words, $\mathbf{z}_{eq}$ is a stationary point of

$$I(\mathbf{z}) = H + \lambda_k C_k. \tag{7.5}$$

If $\mathbf{z}_{eq}$ is an extremum of $I(\mathbf{z})$, then $\mathbf{z}_{eq}$ is a stable equilibrium state because $I(\mathbf{z})$ is a constant of the motion. [Arnol'd's method can also be generalized to include non-Casimir invariants besides the Hamiltonian. See Holm et al. (1985).]

Every equilibrium state $\mathbf{z}_{eq}$ is a stationary point of (7.5) for some choice of constants $\{\lambda_k\}$. For any nearby state $\mathbf{z}_{eq} + \Delta\mathbf{z}$, we define

$$\begin{aligned}\Delta I(\Delta\mathbf{z}; \mathbf{z}_{eq}) &= I(\mathbf{z}_{eq} + \Delta\mathbf{z}) - I(\mathbf{z}_{eq}) \\ &= 1/2\,\partial^2 I(\mathbf{z}_{eq})/\partial z^i \partial z^j \Delta z^i \Delta z^j + O(\Delta z^3).\end{aligned} \tag{7.6}$$

For finitely many degrees of freedom, $\mathbf{z}_{eq}$ is stable to finite size perturbations if the quadratic form in (7.6) is definite. For infinitely many degrees of freedom (the continuum case), the definiteness of this form guarantees only that $\mathbf{z}_{eq}$ is stable in a linear approximation to (7.1). In fact, the quadratic form in (7.6) is a Hamiltonian for the dynamics (7.1) linearized about $\mathbf{z}_{eq}$. To prove *nonlinear* stability of $\mathbf{z}_{eq}$, we seek a definite quadratic form in $\Delta\mathbf{z}$ that is always between $\Delta I(\Delta\mathbf{z}; \mathbf{z}_{eq})$ and zero.

Arnol'd's original example of two-dimensional inviscid flow provides an illuminating illustration of his method. The governing equation is

$$\partial q/\partial t + \partial(\psi, q)/\partial(x, y) = 0, \qquad q = \nabla^2\psi, \tag{7.7}$$

and $\psi = 0$ at the simply connected boundary. The (reduced) phase space is the space of stream functions $\psi(\mathbf{x})$ that are zero on the boundary of the fluid. The general Casimir is

$$C_F[\psi] = \iint d\mathbf{x}\ F(q), \tag{7.8}$$

where $F(q)$ is an arbitrary function of the vorticity q. By analogy with (7.5), the steady equilibrium solution $\psi_{\text{eq}}(\mathbf{x})$ must be a stationary point of

$$I[\psi] = \iint d\mathbf{x}\ \{1/2\nabla\psi \cdot \nabla\psi + F(q)\}. \tag{7.9}$$

Indeed,

$$\delta I = \iint d\mathbf{x}\ \{-\psi + F'(q)\}\,\delta q = 0 \tag{7.10}$$

implies that

$$\psi_{\text{eq}} = F'(q_{\text{eq}}) \equiv \Psi(q_{\text{eq}}). \tag{7.11}$$

The arbitrary function F corresponds to the arbitrary constants λ_k in (7.4). Then we have

$$\begin{aligned}\Delta I(\Delta\psi; \psi_{\text{eq}}) &= I(\psi_{\text{eq}} + \psi) - I(\psi_{\text{eq}}) \\ &= \iint d\mathbf{x}\ \{(\Delta\mathbf{u})^2 + 1/2F''(q_{\text{eq}})(\Delta q)^2\} + O(\Delta^3).\end{aligned} \tag{7.12}$$

Thus, ψ_{eq} is stable in a linear approximation if

$$\Psi'(q_{\text{eq}}) > 0, \qquad \text{all } \mathbf{x}. \tag{7.13}$$

A more thorough analysis (Arnol'd 1966b, Holm et al. 1985) slightly generalizes Rayleigh's classic stability theorem. If, more generally,

$$\Psi'(q_{\text{eq}}) > \text{const} > 0, \qquad \text{all } \mathbf{x}, \tag{7.14}$$

then

$$0 < \Delta I^*(\Delta\psi; \psi_{\text{eq}}) \equiv \iint d\mathbf{x}\ \{(\Delta\mathbf{u})^2 + \text{const}(\Delta q)^2\} < \Delta I, \tag{7.15}$$

and ψ_{eq} is nonlinearly stable in the norm ΔI^*. McIntyre & Shepherd (1987) show that (7.9, 7.11) can be used to rewrite ΔI exactly in a form that emphasizes its $O(\Delta^2)$ size, namely

$$\Delta I(\Delta\psi;\psi_{\text{eq}}) = \iint d\mathbf{x}\ \{1/2\Delta\mathbf{u}\cdot\Delta\mathbf{u}+B(\Delta\psi;\psi_{\text{eq}})\}, \tag{7.16}$$

where

$$B(\Delta\psi;\psi_{\text{eq}}) = \int_0^{\Delta q} \{\Psi(q_{\text{eq}}+q')-\Psi(q_{\text{eq}})\}\ dq'. \tag{7.17}$$

The expression (7.16) and its analogues for quasi-geostrophic flow (McIntyre & Shepherd 1987) are the finite-amplitude generalization of a conserved quantity discovered by Andrews (1983) for small disturbances to steady basic flow.

8. *Reference Flows and Disturbances*

It is often useful to regard fluid motion as the sum of a reference flow, defined mainly for convenience, and an arbitrary disturbance therefrom. Then interest attaches to the interactions between the reference flow and the disturbance. The most useful statements about these interactions take the form of conservation laws. When these conservation laws are derived directly from the conventional Eulerian equations of motion, the manipulations required are often tedious and unrevealing. However, the conservation laws reflect obvious symmetry properties of the fluid Hamiltonian. A Hamiltonian perspective therefore provides the physical motivation for seeking conservation laws, and it leads naturally to their most general formulation. The fundamental papers include those by Eckart (1963), Sturrock (1962), Whitham (1965, 1967), Hayes (1970), Dewar (1970), Bretherton (1971), and Andrews & McIntyre (1978a,b, 1979). However, the following discussion is very closely based on Bretherton (1976). For a recent review, see Grimshaw (1984).

Let

$$\mathbf{x} = \mathbf{x}(\mathbf{a},\tau) \tag{8.1}$$

be the general fluid motion and set

$$\mathbf{x}(\mathbf{a},\tau) = \mathbf{X}(\mathbf{a},\tau)+\boldsymbol{\xi}(\mathbf{X},T), \tag{8.2}$$

where $\mathbf{X}(\mathbf{a},\tau)$ is the reference flow and $\boldsymbol{\xi}(\mathbf{X}(\mathbf{a},\tau),\tau)$ is the displacement at time τ of the fluid particle labeled by $\mathbf{a}$ from the position it would have if it had moved with the reference flow. Here $T=\tau$, but $\partial/\partial T$ will imply that $\mathbf{X}$ is held fixed. Hamilton's principle (2.10) requires that the action be stationary for arbitrary variations in the map (8.1) from $\mathbf{a}$-space into $\mathbf{x}$-space. Substitution of (8.2) into (2.6) yields an expression for the action

$$\iiiint d\tau\, d\mathbf{a}\, \{1/2(\partial\mathbf{X}/\partial\tau + \partial\boldsymbol{\xi}/\partial T + \partial\mathbf{X}/\partial\tau \cdot \nabla\boldsymbol{\xi})^2$$
$$-E(\partial(\mathbf{X})/\partial(\mathbf{a})\, \partial(\mathbf{X}+\boldsymbol{\xi})/\partial(\mathbf{X}))\} \tag{8.3}$$

that depends both on the reference-flow mapping from **a**-space into **X**-space and on the disturbance mapping from **X**-space into $\boldsymbol{\xi}$-space. We have assumed for convenience that the fluid is homentropic.

First suppose that the reference flow is *arbitrarily prescribed.* In particular, $\mathbf{X}(\mathbf{a}, \tau)$ need not itself be a solution of the fluid equations, and in the general case where it is not, it can best be regarded as a field of moving observers. No approximation is involved because the mapping $\boldsymbol{\xi}(\mathbf{X}, T)$ is still completely general. The action (8.3) may be rewritten in the form

$$\iiiint dT\, d\mathbf{X}\, R(\mathbf{X}, T)\{1/2(\mathbf{U}(\mathbf{X}, T) + \partial\boldsymbol{\xi}/\partial T + \mathbf{U}\cdot\nabla\boldsymbol{\xi})^2$$
$$-E(R^{-1}\partial(\mathbf{X}+\boldsymbol{\xi})/\partial(\mathbf{X}))\} = \iiiint \Pi_i\, dX_i\, L(\partial\xi_i/\partial X_j), \tag{8.4}$$

where $(X_0, X_1, X_2, X_3) = (T, X, Y, Z)$ are space-time coordinates, $\nabla \equiv (\partial_X, \partial_Y, \partial_Z)$, and

$$\mathbf{U}(\mathbf{X}, T) = \partial\mathbf{X}/\partial\tau \quad \text{and} \quad R(\mathbf{X}, T) = \partial(\mathbf{a})/\partial(\mathbf{X}) \tag{8.5}$$

are, respectively, the velocity and density fields associated with the reference flow. Hamilton's principle now requires that (8.4) be stationary for variations $\delta\boldsymbol{\xi}(\mathbf{X}, T)$. The reference flow enters (8.4) as a "medium" through the prescribed functions $\mathbf{U}(\mathbf{X}, T)$ and $R(\mathbf{X}, T)$. If this reference flow is independent of one space-time coordinate, say X_r, then (8.4) is invariant to space-time translations in the X_r-direction. To discover the conservation law associated with this symmetry property, we consider variations of the form

$$\delta\xi_i = \partial\xi_i/\partial X_r\, \delta f(X_j), \tag{8.6}$$

where δf is an arbitrary infinitesimal function of the space-time coordinates. For variations (8.6), Hamilton's principle leads directly to

$$\partial T_{rj}/\partial X_j = 0, \tag{8.7}$$

where

$$T_{rj} = \partial L/\partial(\partial\xi_i/\partial X_j)\, \partial\xi_i/\partial X_r - L\delta_{jr} \tag{8.8}$$

is the so-called energy-momentum tensor. Repeated indices are summed. When the reference flow is steady (i.e. $r = 0$), T_{00} is the *pseudo-energy* *density*, and T_{0j} ($j \neq 0$) are its fluxes. When $r \neq 0$, T_{r0} is the *pseudo-momentum density*. The conserved quantity (7.16) is the pseudo-energy corresponding to the steady reference flow ψ_{eq}. Pseudo-momentum conservation laws are also called *generalized Eliassen-Palm theorems* in meteorology.

Now suppose the reference flow is not independent of any space-time coordinate. We can still *create* a conservation law of the type (8.6) by simply introducing an extra independent variable μ into the disturbance field $\boldsymbol{\xi}$. That is, we generalize (8.2) to

$$\mathbf{x}(\mathbf{a}, \tau) = \mathbf{X}(\mathbf{a}, \tau) + \boldsymbol{\xi}(\mathbf{X}, T, \mu) \tag{8.9}$$

and regard the new variable μ as an *ensemble parameter* that identifies the members of a continuous collection of flows. With no loss in generality, we assume that $\boldsymbol{\xi}$ depends periodically on μ with unit period. Then

$$\oint d\mu = \langle \quad \rangle \tag{8.10}$$

is the ensemble average. If the reference flow is defined so that

$$\oint d\mu \; \boldsymbol{\xi}(\mathbf{X}, T, \mu) = 0, \tag{8.11}$$

then the reference flow is really a *mean flow*, and (8.10) is the *generalized Lagrangian mean* introduced by Andrews & McIntyre (1978a).

Now, the mean flow $\mathbf{X}(\mathbf{a}, \tau)$ is, by design, independent of μ. Thus the *averaged action*

$$\oint d\mu \iiiint \Pi_i \, dX_i L \tag{8.12}$$

is invariant to translations in the μ-direction. The analogue of (8.7) is just

$$\partial T_{\mu j}/\partial X_j + \partial T\mu\mu/\partial\mu = 0 \qquad \text{(no summation on } \mu\text{)}, \tag{8.13}$$

and the average of (8.13) is

$$\partial \langle T_{\mu j} \rangle / \partial X_j = 0, \tag{8.14}$$

where

$$\langle T_{\mu j} \rangle = \oint d\mu \; \partial L/\partial(\partial \xi_i/\partial X_j) \; \partial \xi_i/\partial \mu. \tag{8.15}$$

Equation (8.14), which expresses the conservation of *generalized wave* *action* $\langle T_{\mu 0}\rangle$ (Hayes 1970), is simply the field-theory version of the classical action conservation law

$$d/dt \oint d\mu \; p_i(dq_i/d\mu) = 0, \qquad p_i = \partial L(q_j, dq_j/dt)/\partial(dq_i/dt), \tag{8.16}$$

for a discrete system with generalized coordinates $\{q_i(t)\}$. See, for example, Lanczos (1970, pp. 180–83). The integration in (8.16) is around a closed loop of evolving states in phase space. The conservation law (8.16) arises from the invariance of this metasystem to a shift in the system-labeling parameter μ around the loop, in the same way that Kelvin's theorem (4.14) arises from a shift in particle label around a closed material loop of fluid.

With L given by (8.4), (8.14, 8.15) can be brought into the general form given by Andrews & McIntyre (1978b, his Equation 2.15). If, at the other extreme, we assume that ξ in (8.4) takes the very restricted form

$$\xi = \mathbf{A}(\mathbf{X}, T)\cos(\Theta(\mathbf{X}, T) + 2\pi\mu) \tag{8.17}$$

of an *infinitesimal, slowly varying* wave train, then (8.14) reduces to the familiar form

$$\partial/\partial t(E'/\omega') + \nabla\cdot[(\mathbf{U}+\mathbf{C}_g)E'/\omega'] = 0 \tag{8.18}$$

in which E', $\mathbf{C}_g$, and ω' are, respectively, the energy, group velocity, and frequency of the sound wave in a reference frame moving with the mean flow. The result (8.18) also requires the dispersion relation, which can be obtained from (8.4, 8.17) by amplitude variations $\delta\mathbf{A}$. This is essentially Whitham's (1965, 1967) method, although his specific procedure, which would bypass (8.14), is actually more streamlined.

For nongeneral (e.g. infinitesimal) $\xi(\mathbf{X}, T, \mu)$, it is inconsistent to regard the reference flow $\mathbf{X}(\mathbf{a}, \tau)$ as prescribed, and we must vary both the disturbance and the reference flow to satisfy Hamilton's principle. The variations $\delta\mathbf{X}(\mathbf{a}, \tau)$ and $\delta\xi(\mathbf{X}, T, \mu)$ yield coupled equations for the mean flow and disturbance. Since these variations are taken independently, all the above results for the disturbance flow remain valid.

With respect to mean-flow variations $\delta\mathbf{X}(\mathbf{a}, \tau)$, $\xi(\mathbf{X}, T, \mu)$ acts like a prescribed function of $\mathbf{X}(\mathbf{a}, \tau)$. Now suppose that the ensemble average (8.10) is equivalent to an average over space coordinate x_r. Then the μ-averages of $\xi(\mathbf{X}(\mathbf{a}, \tau), \tau, \mu)$, that appear in the averaged Lagrangian are independent of X_r. That is, the statistics of the disturbance field, which comprise the "medium" for the mean flow, are invariant to a translation in the x_r-direction. The resulting conservation law, which can be obtained by considering mean-flow variations $\delta X_r(\mathbf{a}, \tau)$, has been called a "nonacceleration theorem" because it sometimes implies that disturbances cause no permanent change in the mean flow (Andrews & McIntyre 1978a).

9. *Approximate Dynamical Equations*

We have so far entertained only the exact equations for a perfect fluid. However, many commonly used approximate dynamical equations exhibit a Hamiltonian structure analogous to the exact case (and its generalization to include electromagnetic fields). Examples include the reduced magneto-hydrodynamic (MHD) equations (Morrison & Hazeltine 1984) and a four-field extension thereto (Hazeltine et al. 1987), the shallow-water equations (Salmon 1983), the quasi-geostrophic equations (Holm 1986), the Boussinesq surface-wave equations (Whitham 1965), and the Boussinesq internal-wave equations (Benjamin 1986). These examples raise the interesting possibility that Hamiltonian methods can be used to generate useful new approximations. In this spirit, Salmon (1983, 1985) derived a new family of approximations to nearly geostrophic flow, including a generalization of Hoskins' (1975) semi-geostrophic equations to the case of a spatially varying Coriolis parameter. His method has two clear advantages over other perturbation procedures. First, the approximate dynamical equations exactly conserve approximations to the exact invariants of the motion, because the approximations, which are applied directly to the fluid Lagrangian, do not disturb the corresponding symmetry properties. Second, the Hamiltonian perspective suggests transformations to new dependent and independent variables in which the approximate physics takes its simplest mathematical form.

Miles & Salmon (1985) invoke Hamilton's principle for a homogeneous fluid with free surface with the added constraint that the fluid moves in vertical columns, and they thereby obtain a simple, unified derivation of previously known equations for long gravity waves. Their procedure, which is sketched as an example in the remainder of this section, demonstrates that the ideas of Sections 1–6 apply profitably to approximations.

The exact equations for a three-dimensional homogeneous perfect fluid with a free surface and flat bottom result from Hamilton's principle (2.10), with the internal energy replaced by the potential energy associated with the free-surface displacement and by the constraint

$$\partial(x, y, z)/\partial(a, b, c) = 1, \tag{9.1}$$

where the constant mass-density has been taken as unity for convenience. The particle labels $\mathbf{a}$ can always be assigned to correspond to the Cartesian locations $\mathbf{x}$ in a hypothetical initial state in which the fluid is at rest. Then $c = 0$ and $c = H$ describe the bottom at $z = 0$ and the free surface at $z = h$, respectively, where H is the undisturbed depth of the fluid. Miles & Salmon

(1985) introduce the fundamental approximation that the fluid always moves in vertical columns, i.e. that

$$x = x(a, b, \tau), \qquad y = y(a, b, \tau) \tag{9.2}$$

are independent of c. Then (9.1) integrates to

$$z = \partial(a, b)/\partial(x, y)c, \qquad 0 < c < H, \tag{9.3}$$

which can be used to eliminate $z(a, b, c, \tau)$ from the general expressions for the kinetic and potential energy. The resulting Lagrangian,

$$L_{\mathrm{GN}} = 1/2H \iint da\, db\, \{(\partial x/\partial \tau)^2 + (\partial y/\partial \tau)^2 + 1/3\varepsilon(\partial h/\partial \tau)^2 - gh\}, \tag{9.4}$$

depends only on the horizontal locations (9.2) of the vertical fluid columns. In (9.4) g is the gravity constant, and

$$h = \partial(a, b)/\partial(x, y)H \tag{9.5}$$

is the vertical depth of the fluid. The third term in (9.4) is a shallow-water approximation to the vertical kinetic energy. The formal small parameter ε represents the squared ratio of water depth to horizontal length scale, and L_{GN} differs from the exact Lagrangian by an error of $O(\varepsilon^2)$. Hamilton's principle

$$\delta \int L_{\mathrm{GN}}\, d\tau = 0 \tag{9.6}$$

yields the equations

$$\delta \mathbf{x}\text{: } D\mathbf{u}/Dt = -g\nabla h - \varepsilon(3h)^{-1}\nabla(h^2 D^2 h/Dt^2), \tag{9.7}$$

where $\mathbf{u}(x, y, t) = (u, v)$ is the horizontal velocity. A direct application of $\partial/\partial\tau$ to (9.5) yields the exact mass-conservation equation,

$$Dh/Dt + h\nabla \cdot \mathbf{u} = 0. \tag{9.8}$$

Equations (9.7, 9.8) are equivalent to the equations obtained by Green & Naghdi (1976) using a method based upon Cosserat surfaces.

Now (9.4) is unaffected by particle-label variations $\delta\mathbf{a}(x, y, t)$ that leave the Jacobian (9.5) unchanged. By the methods of Section 4, this symmetry property leads to the potential vorticity conservation law

$$D\Pi/Dt = 0, \tag{9.9}$$

where

$$\Pi = [\partial v/\partial x - \partial u/\partial y + 1/3\varepsilon\ \partial(Dh/Dt, h)/\partial(x, y)]/h. \tag{9.10}$$

It is interesting to consider the subclass of solutions to (9.7, 9.8) in which the potential vorticity (9.10) is identically zero. Such flows could arise from very distant sources. By the reasoning of Section 3, the Eulerian equivalent of (9.4–9.6) is

$$\delta \iiint dx\, dy\, dt\, \{h[1/2\mathbf{u}\cdot\mathbf{u}+1/6\varepsilon(h\nabla\cdot\mathbf{u})^2-1/2gh-\zeta Da/Dt$$

$$-\eta Db/Dt]+\phi[\partial h/\partial t+\nabla\cdot(h\mathbf{u})]\} = 0 \tag{9.11}$$

for independent variations δh, $\delta\mathbf{u}$, δa, δb, $\delta\zeta$, $\delta\eta$, and $\delta\phi(x, y, t)$. By the same logic as in Section 5, the use of $Da/Dt = Db/Dt = 0$ to eliminate a, b, ζ, and η from (9.11) gives

$$\delta \iiint dx\, dy\, dt\, h[\partial\phi/\partial t+\mathbf{u}\cdot\nabla\phi-1/2\mathbf{u}\cdot\mathbf{u}-1/6\varepsilon(h\nabla\cdot\mathbf{u})^2+1/2gh] = 0. \tag{9.12}$$

The velocity variation of (9.12) yields

$$\delta\mathbf{u}\colon \mathbf{u} = \nabla\phi+\varepsilon(3h)^{-1}\nabla(h^3\nabla\cdot\mathbf{u}), \tag{9.13}$$

for which the potential vorticity (9.10) is zero for any ϕ. Thus (9.13) is the analogue of potential flow (5.4) for the approximation (9.7, 9.8). However, the velocity (9.13) is not the gradient of a scalar and indeed takes a form that would be very hard to guess from (9.7, 9.8) alone.

Unfortunately, it is very difficult to solve (9.13) for $\mathbf{u}$ to eliminate the velocity from (9.12). However, we can consistently replace (9.13) by

$$\mathbf{u} = \nabla\phi+\varepsilon(3h)^{-1}\nabla(h^3\nabla^2\phi), \tag{9.14}$$

which has the same $O(\varepsilon^2)$ accuracy as (9.13). Substitution of (9.14) into (9.12) yields

$$\delta \iiint dx\, dy\, dt\, h\{\partial\phi/\partial t+1/2(\nabla\phi)^2-1/6\varepsilon(h\nabla^2\phi)^2+1/2gh\} = 0 \tag{9.15}$$

to the same $O(\varepsilon^2)$ accuracy as (9.12). The variational principle (9.15) and resulting canonical form of the Boussinesq equations had been previously given by Whitham (1967).

All the above results apply to long-wavelength motions of arbitrary amplitude. Now consider the restricted case of one-dimensional flow, and let the parameter μ represent the ratio of free-surface displacement $d(x, t)$ to mean water depth H. Then, omitting irrelevant terms and setting $H = g = 1$, we can rewrite (9.15) as

$$\delta \iint dx\, dt\, \{d\partial\phi/\partial t + 1/2(1+\mu d)(\partial\phi/\partial x)^2$$
$$- 1/6\varepsilon(1+\mu d)^3(\partial^2\phi/\partial x^2)^2 + 1/2d^2\} = 0 \qquad (9.16)$$

for variations δd and $\delta\phi(x, t)$. Linearized, nondispersive wave equations result from (9.16) with μ and ε set formally to zero. The change of variables

$$r(x, t) = \int^x (\partial\phi/\partial x + d)\, dx', \qquad s(x, t) = \int^x (\partial\phi/\partial x - d)\, dx' \qquad (9.17)$$

transforms these linear equations into the form

$$\partial r/\partial t + \partial r/\partial x = 0, \qquad \partial s/\partial t - \partial s/\partial x = 0 \qquad (9.18)$$

of uncoupled equations for waves moving in opposite directions. Now substitute the inverse of (9.17), namely

$$\phi = 1/2(r+s), \qquad d = 1/2(\partial r/\partial x - \partial s/\partial x) \qquad (9.19)$$

back into (9.16), assume $\mu = O(\varepsilon)$, neglect $O(\varepsilon^2)$ terms, and, following Broer (1975), set the variable $r(x, t)$ formally to zero. The final step corresponds to the assumption that the motion is slowly varying in the reference frame moving with the s-wave. The result (with $\mu = \varepsilon = 1$) is the variational principle

$$\delta \iint dx\, dt\, \{s_x s_t - (s_x)^2 + 1/4(s_x)^3 + 1/6(s_{xx})^2\} = 0 \qquad (9.20)$$

for (one form of) the Korteweg–de Vries equation (Gardner 1971).

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