Numer. Math. 20, 179—192 (1973)

The Finite Element Method with Lagrangian Multipliers*

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Received January 26, 1972

Summary. The Dirichlet problem for second order differential equations is chosen as a model problem to show how the finite element method may be implemented to avoid difficulty in fulfilling essential (stable) boundary conditions. The implementation is based on the application of Lagrangian multiplier. The rate of convergence is proved.

1. Introduction

The finite element method has become the most successfull approximation method in engineering. There is a variety of detailed approaches based on the finite element method. See e.g. [22 and 17] many others. The central idea of the finite element method is to use different variational principles together with a Galerkin procedure applied to picewise smooth functions.

The notion of a variational method is used today in engineering mostly in the narrow sense that the solution is a stationary point. In other words, the bilinear form which determines the stationarity is often not positive definite. For variational principles used in the theory of elasticity, see e.g. [25].

The finite element method has been studied recently from a theoretical point of view also. Many theorems about convergence, error estimates, etc., have been proved. See e.g. [5, 6, 8, 9, 12, 14, 16, 26].

It has been shown that it is not computationally easy to handle the Dirichlet (essential) boundary condition if the variational principle requires fullfillment of these conditions. Different methods have been developed to avoid this difficulty. See e.g. [3, 10, 14, 18] and others. There is an obvious and classical technique to deal with restrictions in the variational principles. It is the theory of Lagrangian multipliers, applied to the finite element method. Nevertheless application of this idea to the finite element method has not been so far theoretically studied.

We shall analyze here a model problem, but the approach is quite general and may be applied in other cases too. Some interesting cases will be brought out in subsequent papers. Let our problem be to solve the differential equation

$$-\Delta u + u = f \quad \text{on} \quad \Omega \tag{1.1}$$

with boundary conditions

$$u = g \quad \text{on} \quad \Omega^{\bullet}. \tag{1.2}$$

* This paper was supported by the Atomic Energy Commission under Contract No. AEC AT (40-1) 3443/4.

We shall assume that Ω is a bounded domain and that its boundary $\Omega^\bullet$ is sufficiently smooth.

The classical technique is to minimize the quadratic functional

$$F(v) = \int_\Omega \left[\sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i}\right)^2 + v^2\right] dx - 2\int_\Omega f v\, dx \tag{1.3}$$

over all functions satisfying the prescribed boundary condition on $\Omega^\bullet$. By the theory of Lagrange multiplier (see e.g. [19]) the solution creates the stationary point for the (not positive definite) functional

$$F(v, \lambda) = \int_\Omega \left[\sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i}\right)^2 + v^2\right] dx - 2\oint_{\Omega^\bullet} \lambda(v-g)\, ds - 2\int_\Omega v f\, dx. \tag{1.4}$$

In the next chapters we will use this principle and show how to use it in the theory of the finite element method. We shall also show that the rate of convergence of this method is the optimal one.

2. The Principle Notions

Throughout the entire paper R_n be the n-dimensional Euclidian space, $x \equiv (x_1, \ldots, x_n)$, $\|x\|^2 = \sum_{i=1}^{n} x_i^2$ and $dx = dx_1 \ldots dx_n$. Let $\Omega \subset R_n$ be a bounded region and $\Omega^\bullet$ its boundary. We will assume that $\Omega^\bullet$ is infinitely many times differentiable.

Let $L_2(\Omega)$ be the space of square integrable functions u on Ω such that

$$\|u\|^2_{L_2(\Omega)} = \int_\Omega |u|^2 dx < \infty.$$

Some times we shall write $L_2(\Omega) = H^0(\Omega)$. Let $\mathscr{E}(\bar{\Omega})$ be the space of all infinitely many times differentiable functions on Ω and such that all derivatives are continuously extenable on $\Omega^\bullet$, analogously $\mathscr{E}(\Omega^\bullet)$ is the space of all infinitely many times differentiable functions on $\Omega^\bullet$. Furthermore let $\mathscr{D}(\bar{\Omega}) \subset \mathscr{E}(\Omega)$ be the subspace of all functions with compact support in Ω.

Let $l \geqq 1$, l integral. The Sobolev space $H^l(\Omega)$ (resp. $H^l_0(\Omega)$) will be the closure of $\mathscr{E}(\Omega)$ (resp. $\mathscr{D}(\bar{\Omega})$) in the norm $\|\cdot\|_{H^l(\Omega)}$ where

$$\|u\|^2_{H^l(\Omega)} = \sum_{0 \leqq |\alpha| \leqq l} \|D^\alpha u\|^2_{L_2(\Omega)}$$

and

$$D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_i \geqq 0, \quad |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

Now let $\psi_i(x) \geqq 0$, $i = 1, 2, \ldots, \nu$, $x \in R_n$ be functions infinitely times differentiable and such that

$$\sum_{i=1}^{\nu} \psi_i(x) = 1 \quad \text{for} \quad x \in \Omega^\bullet.$$

Further let there be a system of local coordinates $x_i^{[s]}$, $i = 1, \ldots, n$ $s = 1, \ldots, \nu$ and $n-1$ dimensional domains $J_s \subset R_{m-1}$, and a system of functions φ_s, $s = 1, \ldots, \nu$

defined on J_s which are infinitely times differentiable, and which create one to one mappings χ_s of J_s onto $\chi_s(J_s) = \Omega_s^\bullet$ where

$$\Omega_s^\bullet = E\,[(x_1^{[s]}, \ldots, x_{n-1}^{[s]}, \varphi(x_1^{[s]}, \ldots, x_{n-1}^{[s]}) \,|\, (x_1^{[s]}, \ldots, x_{n-1}^{[s]}) \in J_s]$$

and such that

$$E\,[x \in \Omega^\bullet | \psi_s(x) > 0] \subset E\,[(x_1^{[s]}, \ldots, x_{n-1}^{[s]}, \varphi(x_1^{[s]}, \ldots, x_{n-1}^{[s]}) \,|\, (x_1^{[s]}, \ldots, x_{n-1}^{[s]}) \in (J_s)_H]$$

where $H > 0$ and

$$(J_s)_H = E\,[x \in J_s | d(x, J_s^\bullet) > H]$$

with $d(x, J_s^\bullet)$ as the distance of x to $J_s^\bullet$. it is easy to see that such a system of functions ψ_i, φ_i and domains J_s actually exist. Let f be defined on $\Omega^\bullet$. Then the function $f_s = \psi_s f = 0$ everywhere outside of $\Omega_s^\bullet$ and $f_s(\chi_s(x))$ is defined on J_s and has compact support.

Let us introduce the Sobolev spaces on $\Omega^\bullet$. Let $l \geqq 0$ an integer. The Sobolev space $H^l(\Omega^\bullet)$ with norm $\|\cdot\|_{H^l(\Omega)}$ is the space of all functions f defined on $\Omega^\bullet$ and such that

$$\|f\|^2_{H^l(\Omega^\bullet)} = \sum_{s=1}^{\nu} \|f_s\|^2_{H^l(J_s)} < \infty .$$

We have introduced the Sobolev spaces with integral derivatives. For Ω with $\Omega^\bullet \in C^\infty$ we may construct Hilbert scales and obtain spaces with fractional derivatives. See e.g. [20 and 21]. Our notation is in agreement with [21]. These interpolated spaces (resp., their norms) are equivalent with Aronszajn-Slobodeckii norm. See e.g. [23]. For $0 < \alpha = [\alpha] + \sigma$, $0 < \sigma < 1$ and $[\alpha]$ integral we may define

$$\|u\|^2_{H^\alpha(\Omega)} = \|u\|^2_{H^\alpha(\Omega)} + \sum_{|k| = [\alpha]} \|D^k u\|_{H^\sigma(\Omega)} \tag{2.1}$$

where

$$\|u\|^2_{H^\sigma(\Omega)} = \int_\Omega \int_\Omega \frac{(u(t) - u(\tau))^2}{(t-\tau)^{n+2\sigma}}\, dt\, d\tau . \tag{2.2}$$

Analogously we define the norm $\|\cdot\|_{H^\alpha(\Omega^\bullet)}$ which is equivalent with the interpolated norm. For α negative we shall define the space $H^\alpha(\Omega^\bullet)$ as a dual space, namely $H^\alpha(\Omega^\bullet) = (H^{-\alpha}(\Omega^\bullet))^1$. For additional information see [21] p. 35.

Let us mention some theorems which will be useful later.

Theorem 2.1. Let $f \in H^k(\Omega)$, $k > \frac{1}{2}$. Then there exists a trace of the function f on $\Omega^\bullet$ and

$$\|f\|_{H^{k-\frac{1}{2}}(\Omega)} \leqq C \|f\|_{H^k(\Omega)} \text{ [1]} \tag{2.3}$$

where C does not depend on f.

Theorem 2.2. Let $f \in H^k(\Omega)$, $k > \frac{3}{2}$. Then there exists a trace $\partial f / \partial n$ an $\Omega^\bullet$ and

$$\left\| \frac{\partial f}{\partial n} \right\|_{H^{k-\frac{3}{2}}(\Omega^\bullet)} \leqq C \|f\|_{H^k(\Omega)} \tag{2.4}$$

where C does not depend on f. For the proof see [21] p. 47.

1 C will be a generic constant with different values on different places.

The theorems hold only for $k>\frac{1}{2}$ (resp. $k>\frac{3}{2}$) and it is possible to show that the theorem is not valid for $k\leqq\frac{1}{2}$ (resp. $k\leqq\frac{3}{2}$). See e.g. [21] p. 49. Nevertheless let us show that the theorem is true if we restrict the space $H^k(\Omega)$ to a smaller one.

Let $\mathscr{S}(\Omega)\subset H^1(\Omega)$ be the space of all functions which satisfy the equation

$$-\Delta u+u=0$$

in the weak sense; i.e., $\mathscr{S}(\Omega)$ be such a subspace of functions u that

$$\int_\Omega\left[\sum_{i=1}^n\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}+uv\right]dx=0$$

for every $v\in H_0^1(\Omega)$. Now we may prove the next theorem.

Theorem 2.3. Let $u\in\mathscr{S}(\Omega)$. Then we have $\partial u/\partial n\in H^{-\frac{1}{2}}(\Omega^\bullet)$ and

$$\left\|\frac{\partial u}{\partial n}\right\|_{H^{-\frac{1}{2}}(\Omega)}\leqq C\|u\|_{H^1(\Omega)} \tag{2.5}$$

where C does not depend on u.

Proof. Because $\mathscr{S}(\Omega)\cap\mathscr{E}(\bar\Omega)$ is dense in $\mathscr{S}(\Omega)$ (see e.g. [1]) we have to prove the for any $v\in H^{\frac{1}{2}}(\Omega^\bullet)\cap\mathscr{E}(\Omega^\bullet)$ and $u\in\mathscr{S}(\Omega)\cap\mathscr{E}(\bar\Omega)$ we have

$$\left|\oint_{\Omega^\bullet}\frac{\partial u}{\partial n}\,v\,ds\right|\leqq C(u)\|v\|_{H^{\frac{1}{2}}(\Omega^\bullet)} \tag{2.6}$$

with

$$C(u)\leqq C\|u\|_{H^{\frac{1}{2}}(\Omega)}.$$

By an inverse imbedding theorem (see e.g. [21]) there exists a linear mapping μ of $H^{\frac{1}{2}}(\Omega^\bullet)$ into $H^1(\Omega)$ such that $\mu(u)=u$ on $\Omega^\bullet$ and $\|\mu(u)\|_{H^1(\Omega)}\leqq C\|u\|_{H^{\frac{1}{2}}(\Omega^\bullet)}$ where C does not depend on u. For $u\in\mathscr{S}(\Omega)\cap\mathscr{E}(\bar\Omega)$ and $v\in\mathscr{E}(\Omega^\bullet)$ we compute

$$\int_\Omega\left[\sum_{i=1}^n\frac{\partial u}{\partial x_i}\,\frac{\partial\mu(v)}{\partial x_i}+u\mu(v)\right]dx=B(u,v). \tag{2.7}$$

Integrating by parts we obtain

$$B(u,v)=\oint_{\Omega^\bullet}\frac{\partial u}{\partial n}\,v\,ds$$

and we have obviously

$$|B(u,v)|\leqq\|u\|_{H^1(\Omega)}\|\mu(v)\|_{H^1(\Omega)}\leqq C\|u\|_{H^1(\Omega)}\|v\|_{H^{\frac{1}{2}}(\Omega)}.$$

Therefore we have

$$|B(u,v)|=\left|\oint_{\Omega^\bullet}\frac{\partial u}{\partial n}\,v\,ds\right|\leqq C\|u\|_{H^1(\Omega)}\|v\|_{H^{\frac{1}{2}}(\Omega^\bullet)}$$

and this in fact proves our theorem.

Let us mention now some theorems about the Dirichlet and Neumann problems.

Theorem 2.4. Let $f\in H^k(\Omega)$, $k\geqq 0$, $g\in H^l(\Omega^\bullet)$, $l\geqq\frac{1}{2}$. Then there exists in $H^1(\Omega)$ exactly one function u such that u is a weak solution of the equation $-\Delta u+u=f$

with boundary condition $u = g$ on $\Omega^{\bullet}$. Furthermore $u \in H^s(\Omega)$ where $s = \min(k+2, l+\frac{1}{2})$ and

$$\|u\|_{H^s(\Omega)} \leqq C\,[\|f\|_{H^k(\Omega)} + \|g\|_{H^l(\Omega^{\bullet})}] \tag{2.8}$$

where C does not depend on f or g.

Theorem 2.5. Let $f \in H^k(\Omega)$, $k \geqq 0$, $g \in H^l(\Omega^{\bullet})$, $l \geqq -\frac{1}{2}$. Then there exists in $H^1(\Omega)$ exactly function u such that u is a weak solution of the equation $-\Delta u + u = f$ with boundary condition $\partial u/\partial n = g$. Furthermore $u \in H^s(\Omega)$ where

$$s = \min(k+2, l+\tfrac{3}{2})$$

and

$$\|u\|_{H^s(\Omega)} \leqq C\,[\|f\|_{H^k(\Omega)} + \|g\|_{H^l(\Omega^{\bullet})}] \tag{2.9}$$

where C does not depend on f or g.

For the proof of these theorems see [21], p. 203.

We proved Theorem 2.3 for $u \in \mathscr{S}(\Omega)$ but it is possible to generalize it.

Now let $f \in L_2(\Omega)$ and let $\mathscr{S}(\Omega, f)$ be the space of all functions which satisfy the equation

$$-\Delta u + u = f$$

in the weak sense i.e. let $\mathscr{S}(\Omega)$ be such a subspace of function $u \in H^1(\Omega)$ that

$$\int\limits_{\Omega} \left[\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i} + u v\right] dx = \int\limits_{\Omega} f v\, dx$$

for every $v \in H_0^1(\Omega)$. Now the following theorem is valid.

Theorem 2.6. Let $u \in \mathscr{S}(\Omega, f)$. Then we have $\partial u/\partial n \in H^{-\frac{1}{2}}(\Omega)$ and

$$\left\|\frac{\partial u}{\partial n}\right\|_{H^{-\frac{1}{2}}(\Omega^{\bullet})} \leqq C\,[\|u\|_{H^1(\Omega)} + \|f\|_{L_2(\Omega)}] \tag{2.10}$$

where C does not depend on u or f.

Proof. Let $u \in \mathscr{S}(\Omega, f)$. Then there exists a linear mapping χ of $L_2(\Omega)$ into $H^2(\Omega)$ such that $u = v + w$ where

$$w = \chi(f), \qquad \|w\|_{H^2(\Omega)} \leqq C\,\|f\|_{L_2(\Omega)}$$

and

$$v \in \mathscr{S}(\Omega), \qquad \|v\|_{H^1(\Omega)} \leqq C\,[\|u\|_{H^1(\Omega)} + \|f\|_{L_2(\Omega)}].$$

In fact w is a particular solution of the equation $-\Delta w + w = f$ such that $\|w\|_{H^2(\Omega)} \leqq C\,\|f\|_{L_2(\Omega)}$ and we may obviously use for our purpose Theorem 2.4. Combining now Theorem 2.3 with Theorem 2.2, we get our result.

Theorem 2.7. Let $g \in H^{-\frac{1}{2}}(\Omega^{\bullet})$ and u be the weak solution of the Neumann problem for the equation $-\Delta u + u = 0$ on Ω, $\partial u/\partial n = g$ on $\Omega^{\bullet}$ in $H^1(\Omega)$ [see Theorem 2.5]. There exists constants $0 < C_1 < C_2 < \infty$ such that

$$C_1 \oint\limits_{\Omega^{\bullet}} g u\, ds \leqq \|g\|^2_{H^{-\frac{1}{2}}(\Omega)} \leqq C_2 \oint\limits_{\Omega^{\bullet}} g u\, ds \tag{2.11}$$

and

$$\|u\|^2_{H^1(\Omega)} = \oint_{\Omega^\bullet} g u \, ds \tag{2.12}$$

Proof. The statement (2.12) follows immediately from the definition of a weak solution. For $g \in H^0(\Omega^\bullet)$ we have

$$\|g\|_{H^{-\frac{1}{2}}(\Omega^\bullet)} = \sup_{v \in H^{\frac{1}{2}}(\Omega^\bullet)} \frac{\left|\oint_{\Omega^\bullet} g v \, ds\right|}{\|v\|_{H^{\frac{1}{2}}(\Omega)}}. \tag{2.13}$$

Let w be the solution of the equation $-\Delta w + w = 0$ with $w = v$ on $\Omega^\bullet$. Using Theorem 2.4 and 2.5 and the definition of the weak solution, we have

$$\begin{aligned} \frac{\left|\oint_{\Omega^\bullet} g v \, ds\right|}{\|v\|_{H^{\frac{1}{2}}(\Omega^\bullet)}} &\leqq C \frac{\|w\|_{H^1(\Omega)} \|u\|_{H^1(\Omega)}}{\|w\|_{H^1(\Omega)}} \\ &\leqq C \|u\|_{H^1(\Omega)} = C \left[\oint_{\Omega^\bullet} g u \, ds\right]^{\frac{1}{2}}. \end{aligned} \tag{2.14}$$

Therefore, we have

$$\|g\|^2_{H^{-\frac{1}{2}}(\Omega)} \leqq C \oint_{\Omega^\bullet} g u \, ds.$$

Because of the density of $H^0(\Omega^\bullet)$ in $H^{-\frac{1}{2}}(\Omega^\bullet)$, one side of (2.11) is proved.

Let us prove now the other side of the enequality (2.11). We have

$$\left|\oint_{\Omega^\bullet} g u \, ds\right| \leqq \|g\|_{H^{-\frac{1}{2}}(\Omega^\bullet)} \|u\|_{H^{\frac{1}{2}}(\Omega^\bullet)}$$

by definition of the norm $\|g\|_{H^{-\frac{1}{2}}(\Omega^\bullet)}$. Using Theorem 2.5 we get

$$\left|\oint_{\Omega^\bullet} g u \, ds\right| \leqq C \|g\|^2_{H^{-\frac{1}{2}}(\Omega^\bullet)}.$$

We in fact show that

$$C_1 \left|\oint_{\Omega^\bullet} g u \, ds\right| \leqq \|g\|^2_{H^{-\frac{1}{2}}(\Omega^\bullet)} \leqq C_2 \left|\oint_{\Omega^\bullet} g u \, ds\right| \tag{2.15}$$

instead of (2.11). But using (2.12) we see that $\oint_{\Omega^\bullet} g u \, ds \geqq 0$ and therefore (2.15) is identical to (2.11).

Let us introduce now two theorems which will play a fundamental role in the next section.

Theorem 2.8. Let H_1 and H_2 be two Hilbert spaces with scalar products $(\cdot, \cdot)_{H_1}$ and $(\cdot, \cdot)_{H_2}$ resp. Further let $B(u, v)$ be a bilinear form on $H_1 \times H_2$, $u \in H_1$, $v \in H_2$ such that

$$|B(u, v)| \leqq C_1 \|u\|_{H_1} \|v\|_{H_2}, \tag{2.16}$$

$$\sup_{\substack{u \in H_1 \\ \|u\|_{H_1} \leqq 1}} |B(u, v)| \geqq C_2 \|v\|_{H_2}, \tag{2.17}$$

$$\sup_{\substack{v \in H_2 \\ \|v\|_{H_2} \leqq 1}} |B(u, v)| \geqq C_3 \|u\|_{H_1}, \tag{2.18}$$

with $C_2 > 0$, $C_3 > 0$, $C_1 < \infty$. Further let $f \in H_2'$ i.e., let f be a linear functional on H_2. Then there exists exactly one element $u_0 \in H_1$ such that

$$B(u_0, v) = f(v) \tag{2.19}$$

for all $v \in H_2$ and

$$\|u_0\|_{H_1} \leqq \frac{\|f\|_{H_2'}}{C_3}. \tag{2.20}$$

For the proof see [9].

Theorem 2.9. Let the assumptions of the Theorem 2.8 be fulfilled. Further let there be given two linear subspaces (closed) $M_1 \subset H_1$ and $M_2 \subset H_2$ and for every $v \in M_2$, let

$$\sup_{\substack{u \in M_1 \\ \|u\|_{H_1} \leqq 1}} |B(u, v)| \geqq d_2(M_1, M_2) \|v\|_{H_2} \tag{2.21}$$

with $d_2(M_1, M_2) > 0$, and for every $u \in M_1$

$$\sup_{\substack{v \in M_2 \\ \|v\|_{H_2} \leqq 1}} |B(u, v)| \geqq d_3(M_1, M_2) \|u\|_{H_1} \tag{2.22}$$

with $d_3(M_1, M_2) > 0$. Let $f \in H_2'$ be given, and let u_0 denote the element of H_1, such that

$$B(u_0, v) = f(v) \tag{2.23}$$

holds for all $v \in H_2$ [such an element exists and is unique by Theorem 2.8].

Let there exist a $\omega \in M_1$ such that

$$\|u_0 - \omega\|_{H_1} \leqq \vartheta. \tag{2.24}$$

Further, let $\hat{u}_0 \in M_1$ be such that

$$B(\hat{u}_0, v) = f(v) \tag{2.25}$$

for all $v \in M_2$. Then,

$$\|u_0 - \hat{u}_0\|_{H_1} \leqq \left[1 + \frac{C_1}{d_3(M_1, M_2)}\right] \vartheta. \tag{2.26}$$

For the proof see [9].

3. The Finite Element Method — Rate of Convergence in $H^1(\Omega)$

Before we discuss the finite element method, we shall introduce some necessary machinery. Let us introduce $\mathcal{S}_h^{t,k}(\Omega)$ and $\mathcal{S}_h^{t,k}(\Omega^\bullet)$ as one parameter families of functions for all $0 < h < 1$. The linear, finite dimensional system of functions $\mathcal{S}_h^{t,k}(\Omega)$ [resp. $\mathcal{S}_h^{t,k}(\Omega^\bullet)$] will be called a (t, k)-regular system for $t \geqq k \geqq 0$ (resp. $t \geqq k \geqq -\frac{1}{2}$) if:

(1) $\mathcal{S}_h^{t,k}(\Omega) \subset H^k(\Omega)$ [resp. $\mathcal{S}_h^{t,k}(\Omega^\bullet) \subset H^k(\Omega^\bullet)$],

(2) if $f \in H^l(\Omega)$ [resp. $f \in H^l(\Omega^\bullet)$]

then there exists $g \in \mathcal{S}_h^{t,k}(\Omega)$ [resp. $g \in \mathcal{S}_h^{t,k}(\Omega^\bullet)$] such that for any $0 \leqq s \leqq k \leqq l$ [resp. $-\frac{1}{2} \leqq s \leqq k \leqq l$]

$$\|g - f\|_{H^s(\Omega)} \leqq C h^\mu \|f\|_{H^l(\Omega)} \quad [\text{resp. } \|g - f\|_{H^s(\Omega^\bullet)} \leqq C h^\mu \|f\|_{H^l(\Omega^\bullet)}] \tag{3.1}$$

where

$$\mu = \min(l - s, t - s)$$

and C does not depend on s, h or f.

The system $\mathcal{S}_h^{t,k}(\Omega^\bullet)$ will be called strongly (t, k) regular if it is (t, k)-regular and if for every $g \in \mathcal{S}_h^{t,k}$, $-\frac{1}{2} \leq q \leq s \leq k$

$$\|g\|_{H^s(\Omega^\bullet)} \leq C h^{-(s-q)} \|g\|_{H^q(\Omega^\bullet)}$$

and C does not depend on s, h or f.

We have studied the construction of the systems $\mathcal{S}_h^{t,k}(\Omega)$, in [4] and [11] and $\mathcal{S}_h^{t,k}(\Omega^\bullet)$ in [11]. Similarily, since we have introduced the norms $H^s(\Omega^\bullet)$ by transformation to the case of $n-1$ dimensional domains, we may construct strongly (t, k)-regular system $\mathcal{S}_h^{t,k}(\Omega^\bullet)$ using principally the results of [4]. Let us remark that we do not introduce the strongly regular systems $\mathcal{S}_h^{t,k}(\Omega)$. These systems are much more difficult to construct (see [7]). Let us mention that these spaces are finite dimensional.

Now we may explain the finite element method for solving the Dirichlet problem

$$-\Delta u + u = f \quad \text{on} \quad \Omega, \tag{3.2}$$

$$u = g \quad \text{on} \quad \Omega^\bullet \tag{3.3}$$

with $f \in L_2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Omega^\bullet)$. Based on the form (1.4) we may introduce a bilinear form

$$B(u, \lambda; v, \mu) = \int_\Omega \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + u v \right) dx - \int_{\Omega^\bullet} (\lambda v + u \mu)\, ds \tag{3.4}$$

and two functionals

$$F(u, \lambda) = \int_\Omega f u\, dx, \tag{3.5}$$

$$G(u, \lambda) = -\oint_{\Omega^\bullet} g \lambda\, ds. \tag{3.6}$$

Then the stationary point (u_0, λ_0) of (1.4) is obviously such that for every (v, μ), we have

$$B(u_0, \lambda_0; v, \mu) = F(v, \mu) + G(v, \mu). \tag{3.7}$$

Let us define precisely the domain of definition of the bilinear form (3.4). Let us have $H = H_1 = H_2 = H^1(\Omega) \times H^{-\frac{1}{2}}(\Omega^\bullet)$ with the norm $\|u, \lambda\|_H^2 = \|u\|_{H^1(\Omega)}^2 + \|\lambda\|_{H^{-\frac{1}{2}}(\Omega^\bullet)}^2$ and let us show that the bilinear form (3.4) is defined on $H \times H$ and fulfills the conditions (2.16), (2.17) and (2.18) in Theorem 2.8. Let us study first the condition (2.16). Using the imbedding Theorem 2.1, we have

$$|B(u, \lambda; v, \mu)| \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + C\,[\|\lambda\|_{H^{-\frac{1}{2}}(\Omega^\bullet)} \|v\|_{H^1(\Omega)} + \|\mu\|_{H^{-\frac{1}{2}}(\Omega^\bullet)} \|u\|_{H^1(\Omega)}]$$

and therefore we get

$$|B(u, \lambda; v, \mu)| \leq C \|(u, \lambda)\|_H \|v, \mu\|_H \tag{3.8}$$

and therefore (2.16) is proved. Because of symmetry by proving (2.18) we will prove (2.17) too.

So let $(u, \lambda) \in H$ be given. Denote by $w \in H^1(\Omega)$ the solution of the Neumann problem for the differential equation $-\Delta w + w = 0$ with the boundary condition $\partial w/\partial n = \lambda$. Using Theorem 2.5, the function $w \in H^1(\Omega)$ exists and for all $v \in H^1(\Omega)$ we have

$$\int_\Omega \left[\sum_{i=1}^{n} \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_i} + vw\right] dx = \oint_{\Omega^\cdot} \lambda v ds. \tag{3.9}$$

Furthermore we know that

$$\|w\|_{H^1(\Omega)} \leqq C \|\lambda\|_{H^{-\frac{1}{2}}(\Omega)}. \tag{3.10}$$

Using Theorem 2.7, we have

$$C_1 \|\lambda\|^2_{H^{-\frac{1}{2}}(\Omega^\cdot)} \leqq \oint_{\Omega^\cdot} w \lambda ds \tag{3.11}$$

and $0 < C_1$. Let us take $v = u - w$ and $\mu = -2\lambda$. Then obviously,

$$\|(v, \mu)\|_H \leqq C \|(u, \lambda)\|_H. \tag{3.12}$$

Let us show now that

$$B(u, \lambda; v, \mu) \geqq C \|(u, \lambda)\|^2_H. \tag{3.13}$$

If (3.13) holds them using (3.12), we get (2.18). So we have

$$\begin{aligned} B(u, \lambda; v, \mu) = &\int_\Omega \left[\sum \left(\frac{\partial u}{\partial x_i}\right)^2 + u^2\right] dx \\ &- \int_\Omega \left[\sum_{i=1}^{n} \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} + uw\right] dx \\ &- \oint_{\Omega^\cdot} (\lambda u + u\mu) ds + \oint_{\Omega^\cdot} \lambda w ds. \end{aligned} \tag{3.14}$$

Using (3.9), we get

$$B(u, \lambda; v\mu) = \|u\|^2_{H^1(\Omega)} - \oint_{\Omega^\cdot} u(2\lambda + \mu) ds + \oint_{\Omega^\cdot} \lambda w ds. \tag{3.15}$$

Because $2\lambda + \mu = 0$, we obtain (3.13) from (3.11). Therefore we proved the following theorem.

Theorem 3.1. Let $H = H_1 = H_2 = H_1^1(\Omega) \times H^{-\frac{1}{2}}(\Omega^\cdot)$. Then the bilinear form (3.4) satisfies the assumptions of Theorem 2.8.

Now let $f \in L_2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Omega^\cdot)$. Then obviously the functionals (3.5) and (3.6) are continuous. Using Theorem 2.8, we may find $(u_0, \lambda_0) \in H$ such that (3.7) holds for every $(v, \mu) \in H$. Let us investigate the connection between these functions and our original problem (3.2)–(3.3). Let us state it as a theorem.

Theorem 3.2. Let $f \in L_2(\Omega)$, $g \in H^{\frac{1}{2}}(\Omega^\cdot)$ and let $(u_0, \lambda_0) \in H$ be such that (3.7) holds for all $(v, \mu) \in H$. Further let $w \in H^1(\Omega)$ be the weak solution of the Dirichlet problem (3.2), (3.3). Then $u_0 = w$ and $\lambda_0 = \partial w/\partial n$.

Proof. First let us state that there exists a solution of the Dirichlet problem (3.2) and (3.3) by Theorem 2.4. Further, it is obvious that $w \in \mathscr{S}(\Omega, f)$ and therefore $\partial w/\partial n \in H^{-\frac{1}{2}}(\Omega^{\cdot})$ by Theorem 2.6. Let us show now that

$$B\left(w, \frac{\partial w}{\partial n}; v, \mu\right) = \int_{\Omega} f v \, dx - \oint_{\Omega^{\cdot}} g \mu \, ds.$$

Because of Theorem 2.5, the function w is the solution of the Neumann problem for the differential equation $-\Delta u + u = f$ and boundary condition $\partial u/\partial n = \partial w/\partial n$. Therefore

$$\int_{\Omega} \left[\sum_{i=1}^{n} \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_i} + w v\right] dx = \oint_{\Omega^{\cdot}} v \frac{\partial w}{\partial n} ds + \int_{\Omega} f v \, dx.$$

Thus we have

$$B\left(w, \frac{\partial w}{\partial n}; v, \mu\right) = \oint_{\Omega^{\cdot}} v \frac{\partial w}{\partial n} ds - \oint_{\Omega^{\cdot}} \frac{\partial w}{\partial n} v \, ds$$

$$- \oint_{\Omega^{\cdot}} w \mu \, ds + \int_{\Omega} f v \, dx = - \oint_{\Omega^{\cdot}} g \mu \, ds + \int_{\Omega} f v \, dx$$

which is what we wanted to be proved.

To use out bilinear for the finite element approach, let us introduce a subspace $M \subset H$ so that

$$M = \mathcal{S}_{h_1}^{t_1, k_1}(\Omega) \times \mathcal{S}_{h_2}^{t_2, k_2}(\Omega^{\cdot})$$

where $\mathcal{S}_{h_1}^{t_1, k_1}(\Omega)$ is (t_1, k_1)-regular system and $\mathcal{S}_{h_2}^{t_2, k_2}(\Omega^{\cdot})$ is a strongly (t_2, k_2)-regular system. In addition, let $M = M_1 = M_2$. Further let us assume that $k_1 \geqq 1$, $k_2 \geqq \frac{1}{2}$ and furthermore that

$$h_2 \geqq K h_1 \qquad K > 0 \tag{3.16}$$

where K is independent of h_1 and will be determined later. Now we may use Theorem 2.9 which yields the finite element method with Langrage multipliers. We shall seek $(\bar{u}_0, \bar{\lambda}_0) \in M$ such that (3.7) is fulfilled for every $(v, \mu) \in M$. This condition gives a finite number of linear conditions for the element $(\bar{u}_0, \bar{\lambda}_0)$ which is determined by a finite number of parameters. We shall show that the system of linear algebraic equations which we have to solve has, under certain assumptions, exactly one solution. Thus will determine the approximate solution.

Let us prove

Theorem 3.3. For K sufficiently large (see (3.16)) we have in the Theorem 2.9 $d_2(M_1, M_2) = d_3(M_1, M_2) > C > 0$ and C does not depend on h_1.

Proof. The proof will be similar to the proof of Theorem 3.1. Let $(u, \lambda) \in M$ be given. Denote by $w \in H^1(\Omega)$ the solution of the Neumann problem for the differential equation $-\Delta w + w = 0$ and the boundary condition $\partial w/\partial n = \lambda$. Because $\lambda \in \mathcal{S}_{h_2}^{t_2, k_2}(\Omega^{\cdot})$ and because the subspace $\mathcal{S}_{h_2}^{t_2, k_2}(\Omega)$ is strongly regular, we have

$$\|\lambda\|_{H^{\frac{1}{2}}(\Omega^{\cdot})} \leqq \frac{C}{h_2} \|\lambda\|_{H^{-\frac{1}{2}}(\Omega^{\cdot})}. \tag{3.17}$$

Using Theorem 2.5, we have

$$\|w\|_{H^1(\Omega)} \leqq \frac{C C_1}{h_2} \|\lambda\|_{H^{-\frac{1}{2}}(\Omega^\bullet)} \tag{3.18}$$

where C_1 is the constant in (2.9). Using the fact that $\mathcal{S}_{h_1}^{t_1, k_1}(\Omega)$ is (t_1, k_1)-regular we may find $z \in \mathcal{S}_{h_1}^{t_1, k_1}(\Omega)$ such that

$$\begin{aligned} \|w - z\|_{H^1(\Omega)} &\leqq C \|w\|_{H^2(\Omega)} h_1 \\ &\leqq C C_1 C_2 \frac{h_1}{h_2} \|\lambda\|_{H^{-\frac{1}{2}}(\Omega^\bullet)}. \end{aligned} \tag{3.19}$$

We know that

$$\oint_{\Omega^\bullet} w \lambda \, ds \geqq C_3 \|\lambda\|^2_{H^{-\frac{1}{2}}(\Omega^\bullet)}. \tag{3.20}$$

Therefore

$$\begin{aligned} \oint_{\Omega^\bullet} z \lambda \, ds &= \oint_{\Omega^\bullet} w \lambda \, ds + \oint_{\Omega^\bullet} \lambda (z - w) \, ds \\ &\geqq C_3 \|\lambda\|^2_{H^{-\frac{1}{2}}(\Omega^\bullet)} - \eta \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} 0 < \eta &< \|\lambda\|_{H^{-\frac{1}{2}}(\Omega)} \|z - w\|_{H^{\frac{1}{2}}(\Omega^\bullet)} \\ &\leqq \|\lambda\|_{H^{-\frac{1}{2}}(\Omega^\bullet)} C_4 \|z - w\|_{H^1(\Omega)} \leqq C C_1 C_2 C_4 \frac{h_1}{h_2} \|\lambda\|^2_{H^{-\frac{1}{2}}(\Omega)}. \end{aligned}$$

Therefore we have

$$\oint_{\Omega^\bullet} z \lambda \, ds \geqq \|\lambda\|^2_{H^{-\frac{1}{2}}(\Omega)} \left[C_3 - \frac{C C_1 C_2 C_4}{K} \right].$$

Taking K sufficiently large we may obtain

$$C_3 - \frac{C C_1 C_2 C_4}{K} \geqq \frac{C_3}{2} > C_0 > 0 \tag{3.22}$$

and hence

$$\oint_{\Omega^\bullet} z \lambda \, ds \geqq C \|\lambda\|^2_{H^{-\frac{1}{2}}(\Omega^\bullet)}. \tag{3.23}$$

Let us take now

$$v = u - z \quad \text{and} \quad \mu = -2\lambda.$$

Obviously $(v, \mu) \in M$ and the rest of the proof is simple repetition of the remainder of the proof of the Theorem 3.1.

The convergence of the proposed method follows now almost immediately from Theorem 2.9 and the basic properties of (t, k)-regular systems

Theorem 3.4. Let $f \in H^k(\Omega)$, $k \geqq 0$, $g \in H^l(\Omega^\bullet)$, $l \geqq \frac{1}{2}$ and let u_0 be the solution of the Dirichlet problem due to Theorem 2.4. Further, let $\bar{u}_0(h_1, h_2)$ and $\bar{\lambda}_0(h_1, h_2)$ be the approximate solution of the finite element method with Lagrange multiplier (with $k_1 \geqq 1$, $k_2 \geqq \frac{1}{2}$) and K in (3.16) be sufficiently large that Theorem 3.3 holds. Then

$$\begin{aligned} \|\bar{u}_0(h_1, h_2) - u_0\|_{H^1(\Omega)} &+ \left\| \bar{\lambda}_0(h_1, h_2) - \frac{\partial u_0}{\partial n} \right\|_{H^{-\frac{1}{2}}(\Omega^\bullet)} \\ &\leqq C \left[h^{\mu_1} \|f\|_{H^k(\Omega)} + h^{\mu_2} \|g\|_{H^l(\Omega^\bullet)} \right] \end{aligned} \tag{3.24}$$

where

$$\mu_1 = \min \left[t_1 - 1, k + 1, t_2 + \tfrac{1}{2} \right] \tag{3.25}$$

$$\mu_2 = \min \left[t_1 - 1, l - \tfrac{1}{2}, t_2 + \tfrac{1}{2} \right] \tag{3.26}$$

with $h = \max(h_1, h_2)$.

Proof. Using Theorem 2.9 we have to show the existence of $\hat{\omega}=(\hat{u},\hat{\lambda})\in M_1$ such that

$$\|u_0-\hat{u}\|_{H^1(\Omega)}\leqq CQ$$

and

$$\left\|\frac{\partial u_0}{\partial n}-\hat{\lambda}\right\|_{H^{-\frac{1}{2}}(\Omega^\cdot)}\leqq CQ$$

where

$$Q=C\,[h^{\mu_1}\|f\|_{H^k(\Omega)}+h^{\mu_2}\|g\|_{H^l(\Omega^\cdot)}].$$

But this follows immediately from Theorem 2.4, Theorem 2.2 and the basic property of the regularity of the systems $\mathscr{S}_{h_1}^{t_1,k_1}(\Omega)$ and $\mathscr{S}_{h_2}^{t_2,k_2}(\Omega)$.

4. The Rate of Convergence in the Space $L_2(\Omega)$

On the previous section we established the rate of convergence in the space $H^1(\Omega)$ of the approximate solution $\bar{u}_0(h_1,h_2)$ obtained by the finite element method with Lagrange multiplier. In this section we will be interested in the error estimate in the space $L_2(\Omega)$. We shall use—with minor modification—the usual technique in obtaining estimates of this kind.

Let us denote $\varepsilon=\bar{u}_0(h_1,h_2)-u_0$ and $\eta=\bar{\lambda}_0(h_1,h_2)-\partial u_0/\partial n$. By Theorem 3.4 we know that

$$\|\varepsilon\|_{H^1(\Omega)}\leqq Q, \qquad \|\eta\|_{H^{-\frac{1}{2}}(\Omega)}\leqq Q$$

where Q is given by (3.24). Therefore also $\|\varepsilon\|_{L_2(\Omega)}\leqq Q$.

Let V be the solution of the Dirichlet problem for the differential equation $-\Delta V+V=\varepsilon$ with homogeneous boundary condition $V=0$ on $\Omega^\cdot$. Using Theorem 2.4 and 2.2 we have

$$\|V\|_{H^2(\Omega)}\leqq CQ, \tag{4.1}$$

$$\left\|\frac{\partial V}{\partial n}\right\|_{H^{\frac{1}{2}}(\Omega^\cdot)}\leqq CQ, \tag{4.2}$$

and for every $(v,\mu)\in H$ we have

$$B\left(V,\frac{\partial V}{\partial n};v,\mu\right)=\int_\Omega \varepsilon v\,dx.$$

On the other hand, we have $(\varepsilon,\eta)\in H$ and therefore

$$B\left(V,\frac{\partial V}{\partial n};\varepsilon,\eta\right)=\|\varepsilon\|^2_{L_2(\Omega)}. \tag{4.3}$$

But,

$$B(w,\varrho;\varepsilon,\eta)=0 \tag{4.4}$$

for every $(w,\varrho)\in M$ because of the definition of $u(h_1,h_2)$ and $\lambda(h_1,h_2)$. It follows from (4.1), (4.2) and the regularity property that there exists $(z,\xi)\in M$ such that

$$\left\|\left(z-V;\frac{\partial V}{\partial n}-\xi\right)\right\|_H\leqq Ch^\mu\|\varepsilon\|_{L_2(\Omega)} \tag{4.5}$$

where

$$\mu=\min\,[t_1-1,\,1,\,t_2+\tfrac{1}{2}]. \tag{4.6}$$

Because of (4.4), we have using (4.3)

$$\|\varepsilon\|^2_{L_2(\Omega)} = B\left(V - z, \frac{\partial V}{\partial n} - \xi; \varepsilon, \eta\right) \leqq \left\|\left(z - V, \frac{\partial V}{\partial n} - \xi\right)\right\|_H \|\varepsilon, \eta\|_H.$$

But using (4.1) and (4.2) and (4.5), we obtain

$$\|\varepsilon\|^2_{L_2(\Omega)} \leqq C h^{\mu} \|\varepsilon\|_{L_2} Q. \tag{4.7}$$

Therefore

$$\|\varepsilon\|^2_{L_2(\Omega)} \leqq C\,[h^{\mu_1+\mu}\|f\|_{H^k(\Omega)} + h^{\mu_2+\mu}\|g\|_{H^l(\Omega)}] \tag{4.8}$$

where μ_1 is given by (3.25), μ_2 by (3.26) and μ by (4.6). We have proved the following theorem.

Theorem 4.1. Under the assumptions of the Theorem 3.4, (4.8) holds with $\varepsilon = \left(u - \tilde{u}_1(h_1, h_2)\right)$.

References

1. Babuška, I.: The stability of the domain of definition with respect to basic problems of the theory of partial differential equations, especially with respect to the theory of elasticity I, II. Czechoslovak Math. J. 76–105, 165–203 (1961).
2. Babuška, I.: Numerical solution of boundary value problems by perturbed variational principle. Technical Note BN-627 (1969), Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.
3. Babuška, I.: The finite element method for elliptic equations with discontinuous coefficients. Computing **5**, 207–213 (1970).
4. Babuška, I.: Approximation by Hill functions. Commentations Math. Univ. Carolinae, **11**, 387–811 (1970).
5. Babuška, I.: Finite element method for domains with corners. Computing **6**, 264–273 (1970).
6. Babuška, I.: The finite element method for elliptic differential equations. Numerical solution of partial differential equations II. SYNSPADE 1970, edited by B. Hubbard. New York, London: Academic Press 1971 (69–107).
7. Babuška, I.: A remark to the finite element method. Commentations Math. Univ. Carolinae **12**, 367–376 (1971).
8. Babuška, I.: The rate of convergence for the finite element method. SIAM J. Num. Anal. **8**, 304–315 (1971).
9. Babuška, I.: Error bound for finite element method. Num. Math. **16**, 322–333 (1971).
10. Babuška, I.: Finite element method with penalty. Technical Note BN-710 (1971), Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.
11. Babuška, I.: Approximation by Hill functions II. Technical Note BN-708 (1971), Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.
12. Babuška, I.: The finite element method for infinite domains I. Technical Note BN-670 (1970), Institute for Fluid Dynamics and Applied Mathematics, University of Maryland. To appear in Math of Comp.
13. Babuška, I., Rosenzweig, M. B.: A finite element scheme for domains with corners. Technical Note BN-720 (1971), Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.
14. Bramble, J. H., Schatz, A. H.: On the numerical solution of elliptic boundary value problems by least square approximation of the data. Numerical solution of partial differential equations II. SYNSPADE 1970, edited by B. Hubbard. New York, London: Academic Press 1971 (107–133).
15. Bramble, J. H., Zlámal, M.: Triangular elements in the finite element method. Math. of Comp. 809–821 (1970).

16. Fix, G., Strang, G.: A Fourier analysis of the finite element variational method. To appear.
17. Gallagher, R. H.: Trends and directions in the applications of Numerical analysis. ONR Symposium on numerical and computer methods in structural mechanics, University of Ill., September 8–10, 1971.
18. Nitsche, J.: Über Variationsprinzip für Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. To appear.
19. Elsgolc, L. E.: Calculus of variations. London, Paris, Frankfurt: Pergamon Press 1961.
20. Krein, S. G., Petunin, Yu I.: Scales of Banach spaces. Russian Math. Surveys **21**, 85–160 (1966).
21. Lions, J. L., Magenes, E.: Problèmès aux limites non homogènes et applications. Vol. 1, Paris: Dunod 1968.
22. Rashid, Y. R.: On computational methods in solid mechanics and stress analysis. Conf. on the effective use of computers in the nuclear industry, April 21–23, 1969, Knoxville.
23. Slobodeckii, M. I.: Generalized Sobolev spaces and their applications to boundary problems for partial differential equations. Am. Math. Soc. Transl. **21**, 207–275 (1966).
24. Strang, G.: The finite element method and approximation theory. Numerical solution of partial differential equations II. SYNSPADE 1970, edited by B. Hubbard. New York, London: Academic Press 1971 (547–585).
25. Washizu, K.: Variational methods in elasticity and plasticity. Pergamon Press 1968.
26. Zlámal, M.: On the finite element method. Numer. Math. **12**, 394–409 (1968).

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